Decomposing the cube into paths and other combinatorial problems

Joshua Paul Erde

Trinity College

A dissertation submitted to the University of Cambridge
for the degree of Doctor of Philosophy
June 2014

Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text. No part of this dissertation has been submitted for any other qualification.

Abstract

The n-dimensional hypercube is the graph on the vertex set $\{0,1\}^n$ in which two vertices x and y are joined by an edge if they differ in exactly 1 coordinate. Many authors have considered problems of decomposing the edge set of the hypercube into copies of smaller subgraphs, such as cycles or trees. In the case of decomposing the hypercube into paths with k edges there are two obvious necessary conditions that k must satisfy. In the second chapter of this thesis we show that these two conditions are also sufficient, verifying a conjecture of Anick and Ramras.

Finding a knight's tour of the standard 8×8 chessboard is a classical mathematical problem, dating back to the 18^{th} century. More generally, one can consider the movement of a knight on chessboards with arbitrary side length, and also larger dimensions. Schwenk fully classified the 2-dimensional chessboards on which a knight's tour is possible and DeMaio and Mathew did the same for 3-dimensional chessboards. Answering a question of DeMaio and Mathew, in the third chapter we extend these results to fully classify the n-dimensional chessboards on which a knight's tour is possible for general n. We go on to show our methods are useful for constructing tours with more general knight-like chess pieces.

One of the most simple combinatorial games is Tic Tac Toe, a number of generalisations of which have been studied. Of particular interest is the unrestricted n-in-a-row game, where two players take turns claiming points of $\mathbb{Z}^2$, the winner being the first to claim n adjacent points in a row, either vertically, horizontally or diagonally. We consider a variant of this game, suggested by Croft, where the number of points claimed increases by 1 each turn, and so on the t^{th} turn a player claims t points. Croft asked how long it takes to win this game. In the fourth chapter we show that, perhaps surprisingly, the time needed to win this game is $(1 - o(1))n$.

The derivation of a subset A of an infinite group G is the set of elements which appear in the difference set AA^{-1} with infinite multiplicity. Protasov analysed a series of results on the subset combinatorics of groups with regards to this concept, and posed a number of questions. In the fifth chapter we present answers to some of those questions.

Acknowledgements

I would like to take this opportunity to thank a number of people and institutions for their support during the last four years.

Firstly I would like to thank EPSRC for providing funding for my PhD, and DPMMS for financial support during my final two terms. I would also like to thank Ben Barber for providing excellent miniseminars, as well invaluable mathematical discussions on varied topics and Vytautas Gruslys for the same. On a more personal level I would like to thank my parents for their support, as well as James Strawson, Natasha Kingham, Rosie Lintott and Barbara Leeney.

Finally I would like to thank my supervisor, Imre Leader. He has been a constant source of enthusiasm, encouragement and direction.

Contents

Notation

In this section we collect some notation that will be used throughout the thesis.

$$\begin{aligned}[x,y] &= \{x, x+1, \dots, y\};\\ [n] &= [1,n];\\ \mathbb{P}(X) &= \{S \,:\, S \subseteq X\}.\end{aligned}$$

A graph G has vertex set $V(G)$ and edge set $E(G)$. Given two graphs G and H, with vertex sets $V(G)$ and $V(H)$, and edge sets $E(G)$ and $E(H)$ respectively, the *Cartesian product* of G and H, which we will denote by $G \times H$, is the graph with

$$\begin{aligned}V(G\times H) &= \{(g,h) \,:\, g \in V(G), h \in V(H)\}\\ E(G\times H) &= \left\{ \big((g_1,h_1),(g_2,h_2)\big) \,:\, \begin{array}{c} g_1 = g_2 \text{ and } (h_1,h_2) \in E(H) \\ \text{or} \\ h_1 = h_2 \text{ and } (g_1,g_2) \in E(G) \end{array} \right\}.\end{aligned}$$

Given two functions $f, g : \mathbb{N} \to \mathbb{N}$ we say that:

- $f = O(g)$ if there exists some $k > 0$ such that for all sufficiently large n, $f(n) \leq kg(n)$;
- $f = o(g)$ if for every fixed $k > 0$, for all sufficiently large n, $f(n) \leq kg(n)$;
- $f = \Omega(g)$ if there exists some $k > 0$ such that for all sufficiently large n, $f(n) \geq kg(n)$;
- $f = \Theta(g)$ if there exists some $k_1, k_2 > 0$ such that for all sufficiently large n, $k_1 g(n) \leq f(n) \leq k_2 g(n)$.

Given two subsets A and B of a group G we let $AB = \{ab \,:\, a \in A, b \in B\}$. When either A or B is a single element $\{g\}$ we will simply write gB or Ag. In general we will write the operation of an arbitrary group multiplicatively, although when the group is the integers under addition we use additive notation for clarity.

Chapter 1

Introduction

In this introductory chapter we provide an overview of the problems we will consider in the thesis and the results we obtain. Each of the remaining chapters gives a self-contained presentation of one of the four bodies of work described below.

In Chapter 2 we consider a problem of graph decomposition. Given a graph $G = (V, E)$, a natural question to consider is whether the edge set of G can be decomposed into copies of a smaller graph H. An early example of a theorem of this type is:

Theorem (Walecki (see [40])). *The complete graph K_n can be decomposed into Hamiltonian cycles if n is even, and a set of Hamiltonian cycles and a matching if n is odd.*

The *n-dimensional hypercube*, $\mathcal{Q}_n$, is the graph with vertex set $V = \{0,1\}^n$ and edge set $E = \{(x,y) : ||x-y||_1 = 1\}$, that is, x and y are joined by an edge if they differ in exactly 1 coordinate. The hypercube is a fundamental object of study in combinatorics. The problem of decomposing $\mathcal{Q}_n$ into edge-disjoint subgraphs has been considered by many authors. Recently, applications in the theory of parallel processing [39] have motivated problems of decomposing the hypercube into edge-disjoint trees. Fink [23] showed that, given any tree T_n with n edges, we can decompose $\mathcal{Q}_n$ into 2^{n-1} copies of T_n, and independently Ramras [48] and Jacobson, Truszczyński, and Tuza [31] published similar results.

Mollard and Ramras [42] considered the problem of decomposing the hypercube into paths. They noted that if n is odd, and we wish to decompose $\mathcal{Q}_n$ into paths of length k, there are two obvious necessary conditions that k must satisfy. Firstly, since $|E(\mathcal{Q}_n)| = n2^{n-1}$ we must have that k divides $n2^{n-1}$, which we write as $k \mid n2^{n-1}$. Secondly, since $\mathcal{Q}_n$ is n-regular and n is odd, each vertex must be the endpoint of at least one of the paths. Hence we must have at least 2^{n-1} paths, since each path has 2 endpoints. Therefore we must also have that $k \leq n$. Anick and Ramras [2] conjectured:

Conjecture (Anick and Ramras [2]). *Let n be odd and k such that $k \mid n2^{n-1}$ and $k \leq n$. Then $\mathcal{Q}_n$ can be decomposed into paths of length k.*

They were able to show that the conjecture holds for $n < 2^{32}$, a surprisingly high bound, the largest such hypercube being a graph on $2^{2^{32}-1}$ vertices. The main result of this chapter is to show that the conjecture holds for all n.

Theorem. *Let n be odd and k such that $k \mid n2^{n-1}$ and $k \leq n$. Then $\mathcal{Q}_n$ can be decomposed into paths of length k.*

In Chapter 3 we consider the problem of constructing a knight's tours on multi-dimensional chessboards. A *knight's tour* of an $n \times m$ chessboard is a traversal of the squares of the chessboard using only moves of the knight to visit each square once. We say that a knight's tour is *closed* if the last move of the tour returns the knight to its starting position, otherwise the tour is *open*.

The question of the existence of knight's tours has been studied by mathematicians through the ages, both professional and amateur. The earliest known construction of a knight's tour is an example of a closed tour of an 8×8 chessboard given by a 9^{th}century author, al-Adli ar-Rumi [8], even before the modern game of chess had fully developed. The problem was again considered in the 18^{th}century, with early solutions to the knight's tour problem on the standard 8×8 chessboard given by De Moivre, and by Euler [22]. Euler also showed that knight's tours were possible on other sizes of rectangular chessboard. A natural question to ask is, for which n and m does the $n \times m$ chessboard admit a knight's tour (either open or closed)?

In 1978 Cull and Curtins [15] showed that open knight's tours exist on all $n \times m$ ($n \geq m$) chessboards as long as $m \geq 5$ and also that closed knight's tours exist on all $n \times m$ ($n \geq m$) chessboards as long as $m \geq 5$ and one of n or m is even. In 1991 Schwenk fully answered the question for closed tours.

Theorem (Schwenk [50]). *A closed tour of an $n \times m$ ($n \geq m$) chessboard exists if and only if the following conditions hold:*

1) n or m is even;

2) $m \notin \{1, 2, 4\}$;

3) $(n, m) \neq (4, 3)$, $(6, 3)$ or $(8, 3)$.

One can extend the concept of a knight's tour to higher dimensional chessboards, such as a cube (or more generally an n-dimensional cuboid). In this context a knight's move is one which changes one coordinate by ± 1 and a second coordinate by ± 2. Both Stewart

[51] and DeMaio [17] constructed examples of 3-dimensional knight's tours, and in [18] DeMaio and Mathew fully classified the 3-dimensional cuboids on which a chessboard admits a knight's tour.

Theorem (DeMaoi and Mathew [18])**.** *A closed tour of an $n \times m \times p$ $(n \geq m \geq p \geq 2)$ chessboard exists if and only if the following conditions hold:*

1) n, m or p is even;

2) $n \geq 4$;

3) $m \geq 3$.

Stewart [51], DeMaio [17] and DeMaio and Mathew [18] all asked what happened in higher dimensional cuboids. It seems natural to expect, as in the 2-dimensional case, that a closed knight's tour will exist on all sufficiently large higher dimensional cuboids, with at least one side length even. In fact we show more, and in the main result in this chapter we fully classify the n-dimensional cuboids on which a chessboard admits a knight's tour for all $n \geq 3$.

Theorem. *Let $r \geq 3$. A closed tour of an $n_1 \times n_2 \times \ldots \times n_r$ $(n_1 \geq n_2 \geq \ldots \geq n_r \geq 2)$ chessboard exists if and only if the following conditions hold:*

1) Some n_i is even;

2) $n_1 \geq 4$;

3) $n_2 \geq 3$.

Further to this, one can consider the problem of touring chessboards with pieces with more general knight-like movement. For example we call an (α, β)-*knight* a piece whose moves consist of changing one coordinate by $\pm\alpha$ and a second coordinate by $\pm\beta$. An (α, β)-*tour* of a chessboard is a traversal of the squares using only moves of this form. We give some related results, and make some conjectures, about more general knight's tours. In particular we use our methods to reduce the problem of finding closed (α, β)-tours of sufficiently large n-dimensional chessboards to the 2-dimensional case.

In Chapter 4 we consider a problem from the theory of positional games. A *positional game* is a pair $(X, \mathcal{F})$ where X is a set and $\mathcal{F} \subset \mathbb{P}(X)$. We call X the *board*, and the members $F \in \mathcal{F}$ are *winning sets*. The game is played by two players who alternately claim unclaimed points from the board. Given a particular play of that game, the winner is the first player to claim all points from a winning set. If at no point during the game either player achieves this, the game is a draw. We call a game a *first player win* if the first player has a winning strategy, and similarly a *second player win*. If both players have a drawing strategy then we call the game a *draw*.

A classic example of a positional game is that of Noughts and Crosses. Here the board is a 3×3 grid and the winning sets are lines of 3 points, either horizontally, vertically, or diagonally. It is a simple check, performed by most schoolchildren, that this game is a draw. A well known generalisation of Noughts and Crosses is the *n-in-a-row game*. This is a positional game played on $\mathbb{Z}^2$ where the winning sets are any n consecutive points in a row, either horizontally, vertically or diagonally. For $n \leq 4$ it is possible by case checking to show that the n-in-a-row game is a first player win. For $n \geq 8$ it has been shown [56] that the n-in-a-row game is a draw. It is believed that for $n = 5$ the game is a first player win, and a draw for $n \geq 6$.

In this chapter we consider a related game. Following a suggestion of Croft, we will consider a game played on the same board and with the same winning sets as n-in-a-row. However now on the t^{th} turn a player claims t points. So on the first turn the first player claims 1 point, and on the second turn the second player claims 2 points, and on the third turn the first player claims three points, and so on. Unlike the n-in-a-row game this game is never (with perfect play) a draw, since at time n some player will claim n points and so can fully claim a winning set. Since the game is never a draw, for each n, either the first or second player must have a winning strategy. Croft [13] asked the question, how long does it take for that player to win? Clearly some player can win at time n, by claiming an entire winning line, but it may be that it is possible to win significantly quicker than this. The main result of Chapter 4 is that, perhaps surprisingly, neither player can win in time less than $(1 - o(1))n$.

In Chapter 5 we consider some problems on the subset combinatorics of groups. Many combinatorial problems related to subsets of the integers have natural generalizations to arbitrary groups. For example a number of problems in Ramsey theory are concerned with questions of the following type: Given a partition of $\mathbb{Z}^k$ into finitely many sets, must one of the sets contain a subset with certain structural properties? For instance Van der Waerden's Theorem [52] says that whenever we partition $\mathbb{Z}$ into finitely many sets one of the sets must contain arbitrarily large arithmetic progressions. When these properties make reference to the group structure of $\mathbb{Z}^k$, as in Van der Waerden's Theorem, it is natural to consider these problems in a more general setting, by replacing $\mathbb{Z}^k$ with an arbitrary infinite group G.

Some results in this area are concerned with varying notions of the combinatorial size of subsets of an infinite group, see the survery [45]. For example a subset A of G is said to be *large* if there exists a finite subset F of G such that $FA = G$. Given a notion of size, a natural question to ask if, if we partition the group, or a subset of the group, into a finite number of sets, what can we say about the size of these sets? In Chapter 5 we consider some problems of this type, as well as how the varying notions of combinatorial size relate to each other.

For a subset A of an infinite group G we denote by

$$\Delta(A) = \{g \in G : |gA \cap A| = \infty\}.$$

This is sometimes called the *derivation* (or *combinatorial derivation*) of A. We can think of $\Delta(A)$ as the set of elements which appear in the difference set AA^{-1} with infinite multiplicity. In [44] Protasov analysed a series of results on the subset combinatorics of groups with reference to the function Δ, and asked a number of questions. In this chapter we present answers to some of those questions. In particular, Banakh and Protasov showed:

Theorem (Banakh and Protasov [4])**.** *Let G be an infinite group. Given a decomposition $G = A_1 \cup \ldots \cup A_n$ then there exists an i and a subset F of G such that $|F| \leq 2^{2^{n-1}-1}$ and $FA_iA_i^{-1} = G$.*

Noting that $\Delta(A_i) \subset A_iA_i^{-1}$, Protasov [44] asked whether a similar result could hold true for some $\Delta(A_i)$. Our main result in the fifth chapter is that this is indeed the case.

Theorem. *Let G be an infinite group. Given a decomposition $G = A_1 \cup \ldots \cup A_n$ then there exists an i and a subset F of G such that $|F| \leq 2^{2^{n-1}-1}$ and $F\Delta(A_i) = G$.*

The results in Chapter 2 have been submitted for publication. The results in Chapter 3 were submitted to the Electronic Journal of Combinatorics in February 2012. A proof of the main result was also submitted to the same journal at around the same time by Bruno Golénia and Sylvain Golénia. On the advice of the editors we merged the papers and it was published in as “The Closed Knight Tour Problem in Higher Dimensions”, *The Electronic Journal of Combinatorics* 19(4) (2012). The results in Chapter 4 have been submitted for publication. The results in Chapter 5 are in preparation.

Chapter 2

Decomposing the cube into paths

2.1 Introduction

Given a graph $G = (V, E)$, a natural question to consider is whether the edge set of G can be decomposed into edge-disjoint copies of a smaller graph H. If so we say that H *divides* G, which we write as $H \,|\, G$. An early example of a theorem of this type is:

Theorem 1 (Walecki (see [40])). *The complete graph K_n can be decomposed into Hamiltonian cycles if n is even, and a set of Hamiltonian cycles and a matching if n is odd.*

The *n-dimensional hypercube* $\mathcal{Q}_n$ is the graph with vertex set $V = \{0,1\}^n$ and edge set $E = \{(x,y) : ||x-y||_1 = 1\}$, that is x and y are joined by an edge if they differ in exactly 1 coordinate. The hypercube is a fundamental object of study in combinatorics. The problem of decomposing $\mathcal{Q}_n$ into edge-disjoint subgraphs has been considered by many authors. Ringel [49] posed the problem of decomposing $\mathcal{Q}_n$ into Hamiltonian cycles, and showed that this is possible if n is a power of 2. A solution to this problem for general even n can be found in the survey of Alspach, Bermond, and Sotteau [1]. El-Zanati and Eynden [20] extended this result to show that $\mathcal{Q}_n$ can be decomposed into a set of cycles of length d if and only if n is even and $d = 2^s$ for some $s \leq n$, and if n is odd we can decompose $\mathcal{Q}_n$ into a set of cycles of length d and a matching if $d = 2^s$ for some $s \leq n$.

More recently, applications in the theory of parallel processing [39] have motivated problems of decomposing the hypercube into edge-disjoint trees. Fink [23] showed that, given any tree T_n with n edges, we can decompose $\mathcal{Q}_n$ into 2^{n-1} copies of T_n, and independently Ramras [48] and Jacobson, Truszczyński, and Tuza [31] published similar results. Wagner and Mild [53] showed that there exists some specific tree $T_{2^{n-1}}$, with 2^{n-1} edges, such that we can decompose $\mathcal{Q}_n$ into n copies of $T_{2^{n-1}}$, whereas for spanning trees Barden, Davis, Libeskind-Hadas, and Williams [6] showed that, although is not possible to fully partition the edge set into spanning trees, it is possible to decompose $\mathcal{Q}_n$ into $\lfloor \frac{n}{2} \rfloor$ spanning trees and a single path. Bryant, El-Zanati, Eynden, and Hoffman [10] showed that the d star, $K_{1,d}$, divides $\mathcal{Q}_n$ if and only if $d \leq n$ and d divides $|E(\mathcal{Q}_n)| = n2^{n-1}$.

Mollard and Ramras [42], considered the problem of decomposing the hypercube into paths. They noted that if n is odd, and we wish to decompose $\mathcal{Q}_n$ into paths of length k, there are two obvious necessary conditions that k must satisfy. Firstly, since $|E(\mathcal{Q}_n)| = n2^{n-1}$ we must have that k divides $n2^{n-1}$, which we write as $k \mid n2^{n-1}$. Secondly, since $\mathcal{Q}_n$ is n-regular and n is odd, each vertex must be the endpoint of at least one of the paths. Hence we must have at least 2^{n-1} paths, since each path has 2 endpoints. Therefore we must also have that $k \leq n$. Anick and Ramras [2] conjectured:

Conjecture 2 ([2]). *Let n be odd and k such that $k \mid n2^{n-1}$ and $k \leq n$. Then $\mathcal{Q}_n$ can be decomposed into paths of length k.*

They were able to show that the conjecture holds for $n < 2^{32}$, a surprisingly high bound, the largest such hypercube being a graph on $2^{2^{32}-1}$ vertices. The result of this chapter is to show that the conjecture holds for all n.

Theorem 3. *Let n be odd and k such that $k \mid n2^{n-1}$ and $k \leq n$. Then $\mathcal{Q}_n$ can be decomposed into paths of length k.*

In Section 2.2 we provide a proof of Theorem 3 and go on to discuss what can be said about decomposing $\mathcal{Q}_n$ into paths of length k for even n. In Section 2.3 we discuss decomposing $\mathcal{Q}_n$ into arbitrary trees.

2.2 Decomposing the cube

2.2.1 Proof of Theorem 3

A *walk of length* k is a sequence of vertices $x_1, x_2, \ldots, x_{k+1}$, not necessarily distinct, such that for all $1 \leq i \leq k$ $(x_i, x_{i+1}) \in E(\mathcal{Q}_n)$. A *path* is a walk in which each vertex is distinct. We will often define walks and paths by describing their edge sets. We denote by *even vertices* the set of vertices $(q_1, q_2, \ldots, q_n) \in \mathcal{Q}_n$ such that $|\{i : q_i = 1\}|$ is even, and similarly *odd vertices*. It is apparent that $\mathcal{Q}_n$ is a bipartite graph, with the classes being the even and the odd vertices. Two vertices $x, y \in \mathcal{Q}_n$ are *antipodal* if $||x - y||_1 = n$, and we call a path of length n between two antipodal points an *antipodal path*.

Lemma 4. *For any n, $\mathcal{Q}_n$ can be decomposed into antipodal paths of length n.*

Proof. Given a vertex $(q_1, q_2, \dots, q_n) \in \mathcal{Q}_n$ there is a natural antipodal path to consider, that is

$$\begin{aligned}&\{\big((q_1,q_2,\dots,q_n),(q_1+1,q_2,\dots,q_n)\big),\big((q_1+1,q_2,\dots,q_n),(q_1+1,q_2+1,q_3,\dots,q_n)\big),\\&\quad\dots,\\&\quad\big((q_1+1,q_2+1,\dots,q_{n-1}+1,q_n),(q_1+1,q_2+1,\dots,q_{n-1}+1,q_n+1)\big)\}\end{aligned}$$

where addition is taken modulo 2. If we only take the paths beginning at even vertices then we cover each edge exactly once. Indeed if an edge

$$((p_1,p_2,\dots,p_i,\dots,p_n),(p_1,p_2,\dots,p_i+1,\dots,p_n))$$

is in two of these paths, then we must have that $(p_1, p_2, \dots, p_i, \dots, p_n)$ is the i^{th} vertex in one path and $(p_1, p_2, \dots, p_i + 1, \dots, p_n)$ is the i^{th} vertex in the other. However this would imply that the number of 1s in each vector has the same parity, a contradiction. Hence each edge is covered at most once and since there are 2^{n-1} even vertices, and each path has length n, we have covered $n2^{n-1} = |E(\mathcal{Q}_n)|$ edges. $\square$

Since we can decompose $\mathcal{Q}_n$ into paths of length n it is also clear that we can decompose $\mathcal{Q}_n$ into paths of length t for all $t \mid n$ by subdividing these antipodal paths in the natural way. In fact this simple observation achieves more if we consider the structure these paths induce on $\mathcal{Q}_n$.

Lemma 5. *For any n, let t be such that t is odd and $t \mid n$. If $\mathcal{Q}_{\frac{n}{t}}$ can be decomposed into paths of length s then $\mathcal{Q}_n$ can be decomposed into paths of length ts.*

Proof. Let us consider the decomposition of $\mathcal{Q}_n$ into antipodal paths from Lemma 4. Suppose we split each of the paths into $\frac{n}{t}$ paths of length t. We define a graph G on $\{0,1\}^n$ by joining two vertices if there is a path between them, that is, if one of the paths of length t starts at one of the vertices and ends at the other. We claim that G is just a disjoint union of copies of $\mathcal{Q}_{\frac{n}{t}}$. Indeed given a point $(q_1, q_2, \dots, q_n)$ it is adjacent to the points

$$\begin{aligned}&\{(q_1+1,q_2+1,\dots,q_t+1,q_{t+1},\dots,q_n),\\&\ (q_1,q_2,\dots,q_t,q_{t+1}+1,\dots q_{2t}+1,q_{2t+1},\dots,q_n),\\&\ \dots,\\&\ (q_1,q_2,\dots,q_{n-t},q_{n-t+1}+1,\dots,q_n+1)\}.\end{aligned}$$

So if we divide $\{0,1\}^n$ into equivalence classes under the relation $(q_1, q_2, \dots, q_n) \sim (p_1, p_2, \dots, p_n)$

if $(q_1 - p_1, q_2 - p_2, \ldots, q_n - p_n) \in \{(0,0,\ldots,0),(1,1,\ldots,1)\}^{\frac{n}{t}}$ (where $(0,0,\ldots,0)$ and $(1,1,\ldots,1)$ are of length t), we see that G restricted to each equivalence class is isomorphic to $\mathcal{Q}_{\frac{n}{t}}$, and each edge in G is inside one equivalence class.

We use the decomposition of $\mathcal{Q}_{\frac{n}{t}}$ into paths of length s to decompose G into paths of length s, and see that, when considered in $\mathcal{Q}_n$, a path of length s in G is a walk of length ts. More precisely if we have a path $\{(x_1,x_2),(x_2,x_3),\ldots,(x_{s-1},x_s)\}$ in G we know that each edge (x_i,x_j) corresponds to some path of length t in $\mathcal{Q}_n$,

$$P_t^{i,j} = \{(x_i, x_{\{i,j\}_2}), (x_{\{i,j\}_2}, x_{\{i,j\}_3}), \ldots, (x_{\{i,j\}_{t-1}}, x_j)\}.$$

So we have that

$$\begin{aligned} W = \{&(x_1, x_{\{1,2\}_2}), \ldots, (x_{\{1,2\}_{t-1}}, x_2), (x_2, x_{\{2,3\}_2}), \ldots, \\ &(x_{\{2,3\}_{t-1}}, x_3), (x_3, x_{\{3,4\}_2}), \ldots, (x_{\{s-1,s\}_{t-1}}, x_s)\} \end{aligned}$$

is a walk of length ts in $\mathcal{Q}_n$. It remains to check that there are no repeated vertices in W.

Since the decomposition of $\mathcal{Q}_{\frac{n}{t}}$ was into paths we know that $x_1, x_2, \ldots x_t$ are distinct and also we know the form that each $P_t^{i,j}$ takes. Given a vertex interior to a path, say $x_{\{i,j\}_l}$, we know that it agrees with x_i and x_j except in some subset of a block of t consecutive coordinates (specifically differing in the first l, or the last $t-l$ of one of those). Hence given $x_{\{i,j\}_l}$ and the equivalence class of vertices we know x_i and x_j, and so $x_{\{i,j\}_l}$ is interior to only one $P_t^{i,j}$. Since the points in each $P_t^{i,j}$ are distinct, and the interior points are not in the same equivalence class as the endpoints, we have that W has no repeated vertices, and so is a path. $\square$

If n is odd it follows from Lemma 5 that we only need to consider the case of decomposing $\mathcal{Q}_n$ into paths of length 2^r for $2^r \leq n$. Indeed suppose we are given some odd n and a k such that $k \mid n$ and $k \leq n$. Then we have that $k = t2^r$ for some odd t. By Lemma 5 if we can decompose $\mathcal{Q}_{\frac{n}{t}}$ into paths of length 2^r, then we can decompose $\mathcal{Q}_n$ into paths of length k. Note that, since $k \leq n$, we have that $2^r \leq \frac{n}{t}$. To prove this case we will need the following Lemmas:

Lemma 6. *If $\mathcal{Q}_i$ and $\mathcal{Q}_j$ can be decomposed into paths of length k then so can $\mathcal{Q}_{i+j}$.*

Proof. For each vertex $x \in \mathcal{Q}_i$ the subgraph of $\mathcal{Q}_{i+j}$ on the set of vertices

$$\{(q_1, q_2, \ldots, q_{i+j}) : (q_1, q_2, \ldots, q_i) = x\}$$

is isomorphic to $\mathcal{Q}_j$, and so we can decompose each of these, disjoint, subgraphs by using the decomposition of $\mathcal{Q}_j$. Similarly for each vertex $y \in \mathcal{Q}_j$ the subgraph of $\mathcal{Q}_n$ on the set of vertices

$$\{(q_1, q_2, \dots, q_{i+j}) : (q_{i+1}, q_{i+2}, \dots, q_{i+j}) = y\}$$

is isomorphic to $\mathcal{Q}_i$ and so we can decompose these subgraphs by using the decomposition of $\mathcal{Q}_i$. Note that each edge is in exactly one of these subgraphs, since any edge in $\mathcal{Q}_{i+j}$ is between two vertices which differ in exactly one coordinate, which is either in the first i, or the last j. □

Lemma 7. *For all $n \geq 2$, $\mathcal{Q}_n$ contains a Hamiltonian cycle.*

Proof. It is a simple check that $\mathcal{Q}_2$ contains a Hamiltonian cycle, we proceed by induction. Suppose that $\mathcal{Q}_n$ contains a Hamiltonian cycle $\{x_1, x_2, \dots, x_{2^n}\}$, then

$$\{(x_1, 0), (x_2, 0), \dots, (x_{2^n}, 0), (x_{2^n}, 1), (x_{2^n-1}, 1), \dots, (x_2, 1), (x_1, 1)\}$$

is a Hamiltonian cycle in $\mathcal{Q}_{n+1}$. □

We will also need the following folklore result, for a proof see e.g. [1].

Lemma 8. *Let n be even. Then $\mathcal{Q}_n$ can be decomposed into edge-disjoint Hamiltonian cycles.*

Another way to decompose $\mathcal{Q}_n$ into paths, which will inform our method, is as follows. We let X be the set of even vertices in $\mathcal{Q}_n$ and Y be the set of odd vertices. Note that every edge in $\mathcal{Q}_n$ is between X and Y, that is $\mathcal{Q}_n$ is bipartite on the classes X and Y. Given a graph G a *matching* is a set of pairwise non-adjacent edges which meet every vertex.

If we take some matchings $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k$ in $\mathcal{Q}_n$ then we can cover the edges in these matchings by $|X|$ walks of length k, one starting at each vertex in X, by concatenating the matchings in the following way. For example if the edge (x_1, y_{i_1}) is in $\mathcal{M}_1$ and the edge (y_{i_1}, x_{i_2}) is in $\mathcal{M}_2$ and so on then we have that the walk starting at x_1 is $\{(x_1, y_{i_1}), (y_{i_1}, x_{i_2}), \dots (x_{i_{k-1}}, y_{i_k})\}$ if k is odd, and $\{(x_1, y_{i_1}), (y_{i_1}, x_{i_2}), \dots (y_{i_{k-1}}, x_{i_k})\})$ if k is even. To denote the set of walks formed by concatenating $\mathcal{M}_1$ to $\mathcal{M}_k$ in that order, starting at X, we will write

$$\mathcal{W}(\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k, X),$$

and similarly if we start at Y. A pictorial representation of this process is presented in Figure 2.1.

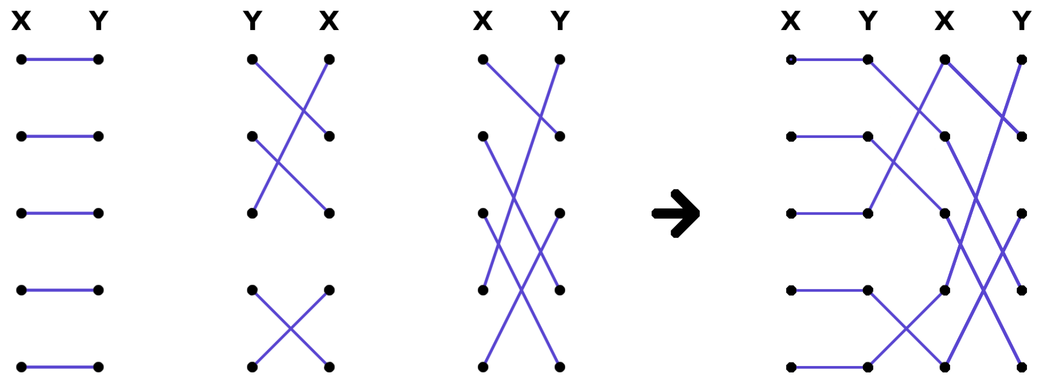

Figure 2.1: Concatenation of 3 matchings, starting at X.

Since $\mathcal{Q}_n$ is n-regular and bipartite, it is a simple consequence of Hall's Theorem [26] that we can decompose the edge set of $\mathcal{Q}_n$ into n perfect matchings. Therefore we can use this method to decompose $\mathcal{Q}_n$ into walks of length k, for any $k \mid n$, by splitting the matchings into sets of size k and concatenating them as above. If we are careful with the matchings we choose and the order we concatenate them in we can ensure that these walks are paths. For example if we take, for $1 \leq i \leq n$, the matchings

$$\begin{aligned}\mathcal{M}_i =&\{((q_1, q_2, \ldots, q_i, \ldots, q_n), (q_1, q_2, \ldots, q_i + 1, \ldots, q_n))\\ &: (q_1, q_2, \ldots, q_i, \ldots, q_n) \in X\},\end{aligned} \tag{2.2.1}$$

where addition is performed modulo 2, we see that $\mathcal{W}(\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n, X)$ is exactly the set of antipodal paths from Lemma 4.

The main idea in the proof of Theorem 3 is to first find a 'small' regular subgraph of $\mathcal{Q}_n$ which will interact nicely with paths we build up from matchings. It will be necessary to treat some small cases by hand and so we will take this opportunity to illustrate the ideas in the method with a small example. For example suppose that for some odd $n \geq 4$ we want to decompose $\mathcal{Q}_n$ into paths of length $2^2 = 4$.

We first claim that, if we want to decompose $\mathcal{Q}_n$ into paths of length 4, without loss of generality we can assume that $n \in [5, 7]$. Indeed if $n \geq 9$ then $n - 5 \geq 4$ and is even and so, by Lemma 8, we have that $\mathcal{Q}_{n-5}$ can be decomposed into cycles of length 2^{n-5}. Since $2^{n-5} > 4$ we can decompose each of these into paths of length 4 and so $\mathcal{Q}_{n-5}$ can be decomposed into paths of length 4. Therefore by Lemma 6 it is sufficient to consider the cases where $n = 5$ or 7. We will just consider the case $n = 5$ in this example.

We view $\mathcal{Q}_5$ as $\mathcal{Q}_3 \times \mathcal{Q}_2$, that is for each $(p_1, p_2) \in \mathcal{Q}_2$ we look at the set of vertices

$(q_1, q_2, q_3, q_4, q_5)$ such that $(q_4, q_5) = (p_1, p_2)$. The induced subgraph of $\mathcal{Q}_5$ on this set of vertices is $\mathcal{Q}_3$. We take a Hamiltonian cycle, C, on $\mathcal{Q}_3$, which exists by Lemma 7, and take the union of these edges over all copies of $\mathcal{Q}_3$. That is, for each $(p_1, p_2) \in \mathcal{Q}_2$ we take the edge set of a copy of C on the subgraph of $\mathcal{Q}_5$ restricted to the vertices $(q_1, q_2, q_3, q_4, q_5)$ such that $(q_4, q_5) = (p_1, p_2)$. We call the union of all these edges G. Note that G is a 2-regular subgraph of $\mathcal{Q}_5$ which covers the vertices of $\mathcal{Q}_5$ with cycles of length 8. Furthermore, since $\mathcal{Q}_3$ is 3-regular, we have that $\mathcal{Q}_3 \setminus C$ is 1-regular and bipartite, that is, it is a matching, $\mathcal{I}^*$. So the union over all copies of $\mathcal{Q}_3$ of $\mathcal{I}^*$, which we will denote by $\mathcal{I}$, is a matching on $\mathcal{Q}_5$. Since we have covered all the edges of each copy of $\mathcal{Q}_3$ with $\mathcal{I}$ and G, we have that the remaining edges of $\mathcal{Q}_5$ are just $\mathcal{M}_4$ and $\mathcal{M}_5$ from Equation 2.2.1. Therefore we have that

$$E(\mathcal{Q}_5) = E(G) \cup E(\mathcal{I}) \cup E(\mathcal{M}_4) \cup E(\mathcal{M}_5).$$

We let $E(G) = E(G^0) \cup E(G^1)$, where G^0 is the restriction of G to the vertices $(q_1, q_2, q_3, q_4, q_5)$ such that $q_4 = 0$ and similarly G^1 is the restriction to the vertices where $q_4 = 1$. This decomposition holds since all the edges of G are contained within copies of $\mathcal{Q}_3$ inside $\mathcal{Q}_5$. We note that both G^0 and G^1 are 2-regular, that is they cover their vertex set with cycles.

We want to use $E(G^0)$ to extend $\mathcal{M}_4$ to paths of length 2. We take each cycle in G^0 and arbitrarily give it an order by labelling the vertices. Given a cycle

$$C = \{(x_1, y_2), (y_2, x_3), (x_3, y_4), \ldots, (x_7, y_8), (y_8, x_1)\}$$

in G^0 we look at the vertices that are matched to $\{x_1, y_2, x_3, y_4, x_5, y_6, x_7, y_8\}$ in $\mathcal{M}_4$. Let us call them $y_1, x_2, y_3, x_4, y_5, x_6, y_7, x_8$ respectively, that is $(x_i, y_i) \in \mathcal{M}_4$ for $1 \leq i \leq 8$. To each edge in the matching we adjoin the 'next' edge in the cycle, that is we form the set of paths

$$\Big\{\{(y_1, x_1), (x_1, y_2)\}, \{(x_2, y_2), (y_2, x_3)\}, \ldots, \{(x_8, y_8), (y_8, x_1)\}\Big\}.$$

(See Figure 2.2).

We repeat this for every cycle in G^0. Let us denote by $\mathcal{P}$ the union of these paths. Note that since G^0 is a graph on exactly half the vertices of $\mathcal{Q}_5$, and no edges in $\mathcal{M}_4$ are between vertices of G^0, we have that each edge of $\mathcal{M}_4$ is used in one of these paths, and also each edge of G^0. Since $\mathcal{Q}_5$ is bipartite we have that each of the paths in $\mathcal{P}$ is between two vertices from X, or two vertices from Y. In fact, moreover, each vertex in X and Y is an endpoint of exactly 1 path, since $\mathcal{M}_4$ covered all the vertices of $\mathcal{Q}_5$ and G^0 was

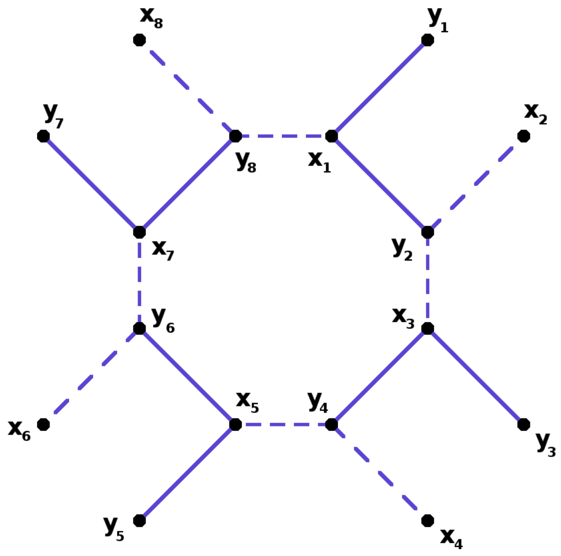

Figure 2.2: Using a cycle to form paths of length 2.

a union of cycles. Let us call the paths between even vertices *even paths*, which we will denote by $\mathcal{P}_e$, and similarly the paths between odd vertices *odd paths*, as $\mathcal{P}_o$. We will use these paths of length 2 to join the edges in a matching into paths of length 4. So we have that

$$E(\mathcal{Q}_5) = E(G) \cup E(\mathcal{I}) \cup E(\mathcal{M}_4) \cup E(\mathcal{M}_5) = E(G^1) \cup E(\mathcal{I}) \cup E(\mathcal{P}) \cup E(\mathcal{M}_5).$$

Since G^1 consists of cycles of length 8, it is simple to decompose it into paths of length 4. We want to use $\mathcal{P}_e$ to connect the edges of $\mathcal{I}$ into paths of length 4, and similarly $\mathcal{P}_o$ for $\mathcal{M}_5$. Each of the even paths is between two vertices in X, and each vertex in X is used exactly once as an endpoint. Therefore we can form walks of length 4 by adding to each path in $\mathcal{P}_e$ the two edges in $\mathcal{I}$ that are connected to its endpoints, see Figure 2.3. This will use each edge in $\mathcal{P}_e$ and $\mathcal{I}$. Similarly we use $\mathcal{P}_o$ to join the edges of $\mathcal{M}_5$ into walks of length 4.

We need to check that the walks we produce in this manner do not repeat any vertices. Given a walk $P = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5)\}$ formed in this way, since each path in $\mathcal{P}$ uses one edge from $\mathcal{M}_4$, we have that without loss of generality either v_1 and v_2 are contained in the set of vertices with $q_4 = 0$ and v_3, v_4 and v_5 in the set of vertices where $q_4 = 1$, or v_1, v_2 and v_3 are contained in the set of vertices with $q_4 = 0$ and v_4 and v_5 in the set of vertices where $q_4 = 1$. Either way we have that P consists of a walk of length 2 in

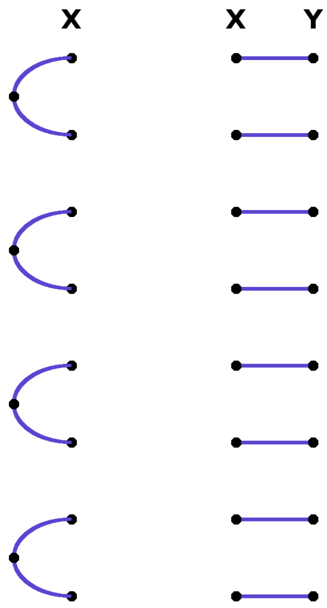

Figure 2.3: $\mathcal{P}_e$ and $\mathcal{I}$.

one subcube and an edge in a disjoint subcube, joined together by an edge between these subcubes. Since the subcubes are disjoint, and each subcube is bipartite, no vertices are repeated and these walks are actually paths. Since we used all the edges of $\mathcal{P}$, $\mathcal{I}$ and $\mathcal{M}_5$ in this process, we have decomposed $\mathcal{Q}_5$ into paths of length 4.

For the general case our idea is similar, we will cover the vertices of a small subcube of $\mathcal{Q}_n$ with some cycles and then decompose the rest of the edges into two sorts of matchings, those contained inside copies of this subcube, like $\mathcal{I}$, and the rest of the form $\mathcal{M}_i$. We combine one of the $\mathcal{M}_i$ with some of the cycles from the subcube to form paths of length 2, and join the rest of matchings into two sets of paths, one starting on X and one starting on Y. We then use the paths of length 2 as before to join the paths starting on X pairwise, and similarly the paths starting on Y. If we have enough $\mathcal{M}_i$ compared to matchings from inside the subcube we can ensure that the walks we produce are actually paths, by making sure that in each walk we never use too many edges from inside the same copy of the small subcube.

Theorem 9. *Let n be odd and r such that $2^r \leq n$. Then $\mathcal{Q}_n$ can be decomposed into paths of length 2^r.*

Proof. Let us first suppose that r is odd. By Lemma 8 it is possible to decompose $\mathcal{Q}_{r+1}$ into Hamiltonian cycles, and so it is possible to decompose it into paths of length 2^r, by splitting each cycle in half. Therefore by Lemma 6 it is sufficient to consider the case where

$n = 2^r + l$ for some odd $1 \leq l \leq r$. Note that $n = 2^r + l \geq r + 2$. We first build a subgraph on $\mathcal{Q}_n$ that is $(l+1)$-regular. For each $(p_1, p_2, \ldots, p_{n-(r+1)}) \in \mathcal{Q}_{n-(r+1)}$ we consider the restriction of $\mathcal{Q}_n$ onto the set of vertices $(q_1, q_2, \ldots, q_n)$ such that $(q_{r+2}, q_{r+3}, \ldots, q_n) = (p_1, p_2, \ldots, p_{n-(r+1)})$, this is isomorphic to $\mathcal{Q}_{r+1}$. By Lemma 8 we can decompose $\mathcal{Q}_{r+1}$ into $\frac{r+1}{2}$ Hamiltonian cycles $C_1, \ldots, C_{\frac{r+1}{2}}$. We split each of the cycles $C_{\frac{l+3}{2}}, \ldots C_{\frac{r+1}{2}}$ into two matchings, so that we have decomposed the edge set of $\mathcal{Q}_{r+1}$ into $\frac{l+1}{2}$ cycles of length 2^{r+1}, $C_1, \ldots, C_{\frac{l+1}{2}}$, and $r - l$ matchings, $\mathcal{I}_1^*, \ldots, \mathcal{I}_{r-l}^*$. For $1 \leq i \leq \frac{l+1}{2}$ we let G_i be the graph formed by taking the union of the edge sets of a copy of C_i on each copy of $\mathcal{Q}_{r+1}$. Similarly for all $1 \leq j \leq r - l$ we let $\mathcal{I}_j$ be the matching formed by taking a copy of $\mathcal{I}_j^*$ on each copy of $\mathcal{Q}_{r+1}$. We now have that

$$E(\mathcal{Q}_n) = \bigcup_{i=1}^{\frac{l+1}{2}} E(G_i) \cup \bigcup_{j=1}^{r-l} E(\mathcal{I}_j) \cup \bigcup_{t=r+2}^{n} E(\mathcal{M}_t).$$

As before we split $E(G_1)$ into $E(G_1^0) \cup E(G_1^1)$ where G_1^0 is the restriction of G_1 to the set of points $(q_1, q_2, \ldots, q_n)$ such that $q_{r+2} = 0$ and G_1^1 is the restriction of G_1 to the set of points $(q_1, q_2, \ldots, q_n)$ such that $q_{r+2} = 1$. Note that both G_1^0 and G_1^1 are 2-regular graphs composed of a disjoint union of cycles of length 2^{r+1}. We combine G_1^0 with $\mathcal{M}_{r+2}$ to form paths of length two as in the previous example. So, for every cycle

$$C = \{(x_1, y_2), (y_2, x_3), (x_3, y_4), \ldots, (x_{2^{r+1}-1}, y_{2^{r+1}}), (y_{2^{r+1}}, x_1)\}$$

in G_1^0 we look at the edges matched to $\{x_1, y_2, \ldots, x_{2^{r+1}-1}, y_{2^{r+1}}\}$ in $\mathcal{M}_{r+2}$. Let us call them $y_1, x_2, \ldots, y_{2^{r+1}-1}, x_{2^{r+1}}$ respectively, that is $(x_i, y_i) \in \mathcal{M}_{r+2}$ for $1 \leq i \leq 2^{r+1}$. To each edge in the matching we adjoin the 'next' edge in the cycle, that is we form the set of paths

$$\Big\{\{(y_1, x_1), (x_1, y_2)\}, \{(x_2, y_2), (y_2, x_3)\}, \ldots, \{(x_{2^{r+1}}, y_{2^{r+1}}), (y_{2^{r+1}}, x_1)\}\Big\}.$$

We repeat this for every cycle in G_1^0, let us denote by $\mathcal{P}$ the union of these paths. As before we split $\mathcal{P}$ into a set $\mathcal{P}_e$ of even paths and a set $\mathcal{P}_o$ of odd sets, and note that every point of X is an endpoint of exactly one path in $\mathcal{P}_e$, and similarly every point in Y is an endpoint exactly one path in $\mathcal{P}_o$.

We use the remaining matchings, $\mathcal{I}_1, \ldots \mathcal{I}_{r-l}$ and $\mathcal{M}_{r+3}, \ldots, \mathcal{M}_n$, to form two sets of walks, one starting at X and one starting at Y, both of length $\frac{n-(l+2)}{2} = \frac{2^r-2}{2} = 2^{r-1} - 1$. We want to order the matchings in such a way that these walks will be paths. For example if we took the set of walks $\mathcal{W}(\mathcal{M}_{r+3}, \mathcal{I}_1, \mathcal{M}_{r+4}, \mathcal{I}_2, \mathcal{M}_{r+5}, \mathcal{I}_3, \ldots, X)$ alternating between using the $\mathcal{M}_i$ and the $\mathcal{I}_j$, at least until we run out of $\mathcal{I}_j$s, then the walks we form will actually be paths. Indeed, if we pick two vertices in the walk x and y which have an edge from $\mathcal{M}_i$, for some i, between them in the walk, then x and y do not agree in the i^{th} coordinate. Therefore the only points that could be repeated in each walk are those joined by an edge in some $\mathcal{I}_j$, but clearly these are distinct, since $\mathcal{Q}_n$ has no loops. So we want to have at least as many $\mathcal{M}_i$s as we do $\mathcal{I}_j$s, that is we need that $n-(r+2) \geq r-l$. Since $n = 2^r + l$ we need $2^r + 2l - 2 \geq 2r$ and since $l \geq 1$ it is sufficient that $2^r \geq 2r$, which holds for all odd r.

So we form our two sets of paths in this way, one starting at X and one starting at Y, and we use $\mathcal{P}_e$ to join the ones starting at X and $\mathcal{P}_o$ to join the ones starting at Y, as before, into walks of length $2(2^{r-1}-1)+2 = 2^r$. Again it is a simple check that these walks are in fact paths. Let us consider one of the walks formed by combining $\mathcal{W}(\mathcal{M}_{r+3}, \mathcal{I}_1, \mathcal{M}_{r+4}, \mathcal{I}_2, \mathcal{M}_{r+5}, \mathcal{I}_6, \ldots, X)$ and $\mathcal{P}_e$. It consists of two paths from $\mathcal{W}(\mathcal{M}_{r+3}, \mathcal{I}_1, \mathcal{M}_{r+4}, \mathcal{I}_2, \mathcal{M}_{r+5}, \mathcal{I}_6, \ldots, X)$, joined together by a path of length two from $\mathcal{P}_e$. Since an edge of $\mathcal{M}_{r+2}$ was used in each path in $\mathcal{P}_e$ we have that the 2^{r-1} vertices in first path differ from the 2^{r-1} vertices in the second path in the $(r+2)$nd coordinate, and so they are all distinct. Finally the vertex in the middle of the path of length two differs from all of the vertices except its immediate neighbours in the $(r+3)^{\text{th}}$ coordinate, and since those three vertices were in a path in $\mathcal{P}_e$, it is distinct from those two as well.

So, to conclude, we have decomposed $\mathcal{Q}_n$ into some graphs $G_1^1, G_2, G_3 \ldots G_{\frac{l+1}{2}}$ which are each a union of cycles of length 2^{r+1} and a collection of paths of length 2^r, therefore $\mathcal{Q}_n$ can be decomposed into paths of length 2^r.

The case where r is even is similar. Since we can decompose $\mathcal{Q}_{r+2}$ into Hamiltonian cycles it is sufficient to consider the case $n = 2^r + l$ for some odd $1 \leq l \leq r+1$. We view $\mathcal{Q}_n$ as $\mathcal{Q}_{r+2} \times \mathcal{Q}_{n-(r+2)}$ and use the decomposition of $\mathcal{Q}_{r+2}$ into Hamiltonian cycles to split $\mathcal{Q}_n$ into G_is, $\mathcal{I}_j$s and $\mathcal{M}_t$s as in the odd case.

There are two small differences, firstly in order to make the paths of length 2 we need that $\mathcal{M}_{r+3}$ exists. That is we need $n = 2^r + l \geq r+3$, but this holds for all even r, $1 \leq l \leq r+1$. The second difference comes when we want to check that we have at least as many $\mathcal{M}_i$s as $\mathcal{I}_j$s, since now we need that $n-(r+3) \geq r-l+1$, that is $2^r + 2l - 4 \geq 2r$. This holds for all $r \geq 4$, $1 \leq l \leq r+1$, and also for $r = 2, l = 3$. The only remaining case to check is therefore when $r = 2$ and $l = 1$, that is, we need to demonstrate a decomposition of $\mathcal{Q}_5$ into paths of length 4, which we did in the preceding example. □

Proof of Theorem 3. Given $k \,|\, n2^{n-1}$ we have that $k = t2^r$ for some odd $t \,|\, n$. Since $k = t2^r \leq n$ we have that $2^r \leq \frac{n}{t}$, and so by Theorem 9 $\mathcal{Q}_{\frac{n}{t}}$ can be decomposed into paths of length 2^r. Therefore by by Lemma 5 $\mathcal{Q}_n$ can be decomposed into paths of length k. $\square$

2.2.2 Decomposing $\mathcal{Q}_n$ into paths for n even

The case where n is even seems different. For example in the odd case the problem seems just as difficult if we ask for walks instead of paths. However for even n, since every vertex has even degree, $\mathcal{Q}_n$ has an Eulerian cycle, and so it is possible to decompose $\mathcal{Q}_n$ into walks of length k for every $k \mid n2^{n-1}$. If we want to decompose $\mathcal{Q}_n$ into **paths** of length k we still need that $k \mid n2^{n-1}$, but, since the vertices of $\mathcal{Q}_n$ don't have odd degrees, we no longer require that $k \leq n$. For example by Lemma 8 we can decompose $\mathcal{Q}_n$ into paths of length 2^{n-1}, so a more natural condition would seem to be $k < 2^n$, since no path can be longer than $|\mathcal{Q}_n|$.

The methods of Section 2 prove some results towards this. Let P_k be the path of length k, that is with k edges. Since Lemma 5 holds for general n we know that if $n = t2^r$ with t odd then, since by Lemma 8 $\mathcal{Q}_{\frac{n}{t}}$ has a decomposition into paths of length $2^{\frac{n}{t}-1}$, we can decompose $\mathcal{Q}_n$ into paths of length $t2^{\frac{n}{t}-1}$. However, if it were true that $P_k \mid \mathcal{Q}_n$ for all $k \mid n2^{n-1}$ and $k < 2^n$ then we could decompose $\mathcal{Q}_n$ into paths of length $t2^{n-\lceil \log_2 t \rceil}$, since this is the largest power of 2 with $t2^{n-\lceil \log_2 t \rceil} < 2^n$, note the inequality is strict since t is odd and so $\log_2 t$ is not an integer. For example consider $\mathcal{Q}_6$. We have that $3 \mid 6$, and by Lemma 4 we can decompose $\mathcal{Q}_6$ into paths of length 6. However since $3.2^r \mid 6.2^5 = |E(\mathcal{Q}_6)|$ and $3.2^r < 2^6 = |\mathcal{Q}_6|$ for $r \leq 4$ it might be possible to decompose $\mathcal{Q}_6$ into paths of length $3.2^4 = 48$.

Question 10. *Let n be even and k such that $k \mid n2^{n-1}$ and $k < 2^n$. Can $\mathcal{Q}_n$ be decomposed into paths of length k?*

2.3 Decomposing the cube into trees

As mentioned in the introduction, many authors have considered the question of decomposing $\mathcal{Q}_n$ into trees. A natural question to consider is:

Question 11. *Given n, for which trees T can $\mathcal{Q}_n$ be decomposed into copies of T?*

In the case where n is odd there are very natural classifications for the paths and stars which divide $\mathcal{Q}_n$. Indeed, we proved in Section 2.2 that $P_k \mid \mathcal{Q}_n$ if and only if $k \mid n2^{n-1}$ and $k \leq n$ and similarly Bryant, El-Zanati, Eynden, and Hoffman [10] showed that that

$K_{1,d} \,|\, \mathcal{Q}_n$ if and only if $d \,|\, n2^{n-1}$ and $d \leq n$. It might be tempting to conjecture that a similar condition is necessary for all trees, however the conditions $k \leq n$ and $d \leq n$ come from very different considerations. The first since each vertex in $\mathcal{Q}_n$ has odd degree, and the second since each vertex in $\mathcal{Q}_n$ has degree less than n. If we considered a tree with k edges which had many leaves, but maximum degree less than n it might be possible to decompose an odd cube $\mathcal{Q}_n$ into copies of that tree even if $k > n$, and in fact we will show later in this section that this can happen.

Similar ideas as in Section 2.2 can be used to approach this problem. Fink [23] showed that, given any tree T_n with n edges, $T_n \,|\, \mathcal{Q}_n$. The proof is a similar idea to the concatenation of matchings in Section 2.2, and can easily by adapted to cover the case of trees with k edges where $k \,|\, n$.

Proposition 12. *For any n, let k be such that $k \,|\, n$ and let T_k be any tree with k edges. Then $T_k \,|\, \mathcal{Q}_n$.*

Proof. We start by picking a root of the tree v, and labelling the edges of the tree $e_1, e_2, \ldots, e_k$. We decompose $\mathcal{Q}_n$ into the matchings $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$, as in Section 2.2 and split these into sets of k matchings. Given a set of k such matchings, without loss of generality $\mathcal{M}_1, \mathcal{M}_2, \ldots \mathcal{M}_k$, we assign each matching to an edge of the tree, in this case e_i to $\mathcal{M}_i$. We will use these matchings to build $|X|$ copies of T_k, each one rooted at a different vertex in X, where in each copy the edge e_i comes from the matching $\mathcal{M}_i$. Here, as before, X is the set of even vertices in $\mathcal{Q}_n$.

Given $q \in X$ we define a copy of T_k as follows, we map v to q and, for all $w \in T_k$ we note that there is a unique path from v to w in T_k, say along the edges $e_{i_1}, e_{i_2}, \ldots e_{i_r}$. So we can map w to the vertex reached by following the path starting at q in the set $\mathcal{W}(\mathcal{M}_{i_1}, \mathcal{M}_{i_2}, \ldots \mathcal{M}_{i_r}, X)$. We end up with a subset $\{q_v : v \in T_k\} \subset \mathcal{Q}_n$ and it is a simple check that the subgraph of $\mathcal{Q}_n$ induced on these vertices is isomorphic to T_k.

If we repeat this for each $q \in X$ this will use $k|X|$ edges, so we just need to check no edge will be used twice in this construction. However given an edge $(q_1, q_2) \in \mathcal{M}_i$ there is only one possible root whose tree it could be used in. Indeed, given that (q_1, q_2) is in some copy of T_k, we know it corresponds to the edge e_i. Since T_k is bipartite we can split it's vertices into two classes, one containing the root v and one not, note that an endpoint of e_i lies in each class. Since each vertex in the first class must have been mapped to an even vertex, and vice versa, we can identify the vertices of T_k which q_1 and q_2 were mapped to, and hence, by considering the unique path between these vertices and v in T_k, determine the root of the copy of T_k.

For each $0 \leq j \leq \frac{n}{k} - 1$ we repeat this construction using the sets of matchings

$\mathcal{M}_{jk+1}, \mathcal{M}_{jk+2}, \ldots, \mathcal{M}_{(j+1)k}$, and in this way decompose $\mathcal{Q}_n$ into copies of T_k. □

Also Lemma 6 holds in a much more general context.

Lemma 13. *Given graphs G_1 and G_2, suppose that $H \,|\, G_1$ and $H \,|\, G_2$. Then $H \,|\, G_1 \times G_2$.*

Proof. As in Lemma 6 for each vertex $v \in G$ the subgraph of $G_1 \times G_2$ on the set of vertices $\{(v, w) : w \in G_2\}$ is isomorphic to G_2, and so we can decompose each of these, disjoint, subgraphs into copies of H by using the decomposition of G_2. Similarly for each vertex $w \in G_2$ the subgraph of $G_1 \times G_2$ on the set of vertices $\{(v, w) : v \in G_1\}$ is isomorphic to G_2 and so we can decompose these subgraphs by using the decomposition of G_2. Note that each edge is in exactly one of these subgraphs, since any edge in $G_1 \times G_2$ is either of the form $\big((v_1, w_1), (v_2, w_2)\big)$ such that $v_1 = v_2$ $(w_1, w_2) \in E(G_2)$ or such that $w_1 = w_2$ $(v_1, v_2) \in E(G_1)$. □

Since it is equivalent to Question 11 to determine for each tree T_k which $\mathcal{Q}_n$ it divides, we might hope to show T_k divides a suitable class of $\mathcal{Q}_n$ by an inductive argument as in Section 2.2, using Proposition 12 and Lemma 13. However there are a few ways in which Theorem 3 was significantly simpler. Firstly, whilst for paths we can view a path of length k as a 'path' of length t where each edge corresponds to a path of length $\frac{k}{t}$, for trees there is not in general a natural way to reduce them in this way to smaller trees of a given order, so there doesn't seem to be an obvious extension of Lemma 5. Secondly, once we'd used Lemma 5 to reduce Theorem 3 to the case of paths of length 2^r, Lemma 8 not only allowed us to use $\mathcal{Q}_{r+1}$ or $\mathcal{Q}_{r+2}$ in our inductive step rather than $\mathcal{Q}_{2^r}$, which would have greatly increased the number of base cases to consider, it also gave us a decomposition of $\mathcal{Q}_{r+1}$ or $\mathcal{Q}_{r+2}$ into a number of regular subgraphs, that is the Hamiltonian cycles, that themselves could be decomposed into paths. This allowed us to use only some of them in our constructions, and still have a graph that we could decompose into matchings left. However the trees constructed in Proposition 12 do not seem to be as flexible in this regard.

To conclude the chapter we consider some some specific examples of trees and what can be said about which $\mathcal{Q}_n$ they divide. The only tree with one edge, $P_1 \cong K_{1,1}$, divides $\mathcal{Q}_n$ for all n trivially. Similarly the only tree with two edges, $P_2 \cong K_{1,2}$, divides $\mathcal{Q}_n$ for all $n \geq 2$ by Theorem 3 and Lemma 8, and doesn't divide $\mathcal{Q}_1$. There are two tree with three edges, P_3 and $K_{1,3}$, and by Theorem 3 and the result of Bryant, El-Zanati, Eynden, and Hoffman [10] they both divide $\mathcal{Q}_n$ if and only if n is divisible by 3.

The first interesting case is $k = 4$, here there are three trees with four edges, P_4, $K_{1,4}$ and H (see Figure 2.4). By Lemma 8 P_4 divides $\mathcal{Q}_n$ for all $n \geq 4$, and it doesn't divide

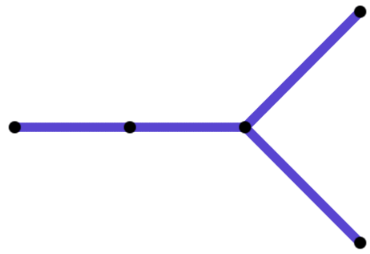

Figure 2.4: The tree H.

$\mathcal{Q}_3$ or $\mathcal{Q}_1$ by Theorem 3, or $\mathcal{Q}_2$ by inspection. Similarly $K_{1,4}$ divides $\mathcal{Q}_n$ if and only if $n \geq 4$ by the result of Bryant, El-Zanati, Eynden, and Hoffman [10]. Similarly H cannot divide $\mathcal{Q}_1$ or $\mathcal{Q}_2$, however we see from Figure 2.5 that $H \mid \mathcal{Q}_3$. Also by Proposition 12 we have that $H \mid \mathcal{Q}_4$. So if we could exhibit a decomposition of $\mathcal{Q}_5$ into copies of H then, by using Lemma 13 and the fact that $\mathcal{Q}_{i+j} = \mathcal{Q}_i \times \mathcal{Q}_j$, we would have that $H \mid \mathcal{Q}_n$ for all $n \geq 3$.

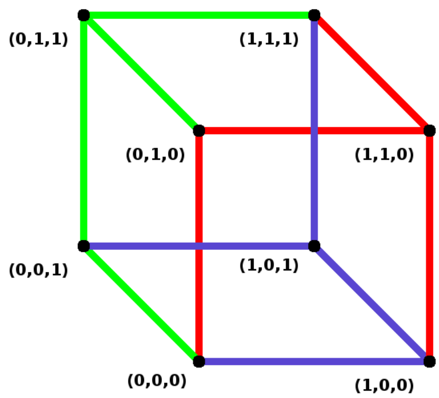

Figure 2.5: A decomposition of $\mathcal{Q}_3$ into copies of H.

We do so as follows. We consider $\mathcal{Q}_5$ as $\mathcal{Q}_3 \times \mathcal{Q}_2$ and in the copies of $\mathcal{Q}_3$ where $(q_4, q_5) \in \{(0,0),(1,1)\}$ we take the copy of H formed by the blue edges in Figure 2.5. We split the remaining edges into four paths of length two $\{(0,0,0),(0,1,0),(1,1,0)\}$, $\{(1,0,0),(1,1,0),(1,1,1)\}$, $\{(0,1,1),(0,0,1),(0,0,0)\}$, $\{(1,1,1),(0,1,1),(0,1,0)\}$. We add to each of these paths the two remaining edges adjacent to their first vertex in $\mathcal{Q}_5$ to form a copy of H, so for example using the first of these paths in the subcube where $(q_4, q_5) = (0,0)$ the copy of H we form is the induced subgraph of $\mathcal{Q}_5$ on the vertex set

$$\{(0,0,0,0,0),(0,1,0,0,0),(1,1,0,0,0),(0,0,0,1,0),(0,0,0,0,1)\}$$

If we look at the edges of $\mathcal{Q}_5$ we have covered so far with copies of H, it is a simple check that this subgraph is isomorphic to its complement in $\mathcal{Q}_5$, and so we can decompose the whole of $\mathcal{Q}_5$ into copies of H.

Here we have in some way exploited the fact that H is composed of two paths of length 2. For general trees it might be useful to consider the various ways in which we can consider them at being composed of smaller trees.

The results in this chapter have been submitted for publication.

Chapter 3

Knight's tours in higher dimensions

3.1 Introduction

A *knight's tour* of an $n \times m$ chessboard is a traversal of the squares of the chessboard using only moves of the knight to visit each square once. A knight's tour is *closed* if the last move of the tour returns the knight to its starting position, otherwise the tour is *open*.

In graph theoretical terms we can consider the movement of a knight on an $n \times m$ chessboard as a graph on the grid of points $[n] \times [m] \subset \mathbb{Z}^2$, the *knight's graph* $K(n, m)$, where each point is joined to all points a knight's move away. That is, $K(n, m)$ is the graph G where $V(G) = \{(x_1, x_2) \,:\, 1 \leq x_1 \leq n \,, 1 \leq x_2 \leq m\}$ and $\big((a_1, a_2), (b_1, b_2)\big) \in E(G) \Leftrightarrow (a_1 - b_1, a_2 - b_2) \in \{(\pm 1, \pm 2)\,,\,(\pm 2, \pm 1)\}$. So an open knight's tour of an $n \times m$ chessboard is a Hamiltonian path in $K(n, m)$, and a closed knight's tour is a Hamiltonian cycle. In this chapter we will mainly be discussing closed tours and so, when the context is clear, we will refer to a closed tour of an $n \times m$ chessboard as simply an $n \times m$ *tour*.

The question of the existence of knight's tours has been studied by mathematicians through the ages, both professional and amateur. The earliest known construction of a knight's tour is an example of a closed tour of an 8×8 chessboard given by a 9^{th}century author, al-Adli ar-Rumi [8] (see Figure 3.1), even before the modern game of chess had fully developed. The problem was again considered in the 18^{th} century, with early solutions to the knight's tour problem on the standard 8×8 chessboard given by De Moivre, and by Euler [22]. Euler also showed that knight's tours were possible on other sizes of rectangular chessboard. A natural question to ask is, for which n and m does the $n \times m$ chessboard admit a knight's tour (either open or closed)?

Multiple authors showed partial results to this question. For example Kraitchik [38]

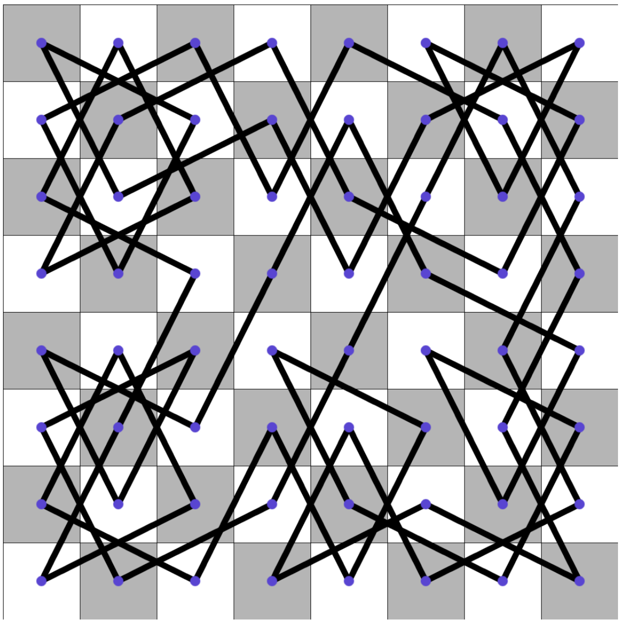

Figure 3.1: al-Adli ar-Rumi's 8×8 knight's tour.

showed that an open $n \times n$ tour exists for all $n \equiv 1 \pmod 4$, and Dudeney [19] claimed without proof that an open $n \times n$ tour exists for all n. In 1978 Cull and Curtins [15] showed that open knight's tours exist on all $n \times m$ $(n \geq m)$ chessboards as long as $m \geq 5$ and also that closed knight's tours exist on all $n \times m$ $(n \geq m)$ chessboards as long as $m \geq 5$ and one of n or m is even. In 1991 Schwenk fully answered the question for closed tours.

Theorem 14 (Schwenk [50])**.** *A closed $n \times m$ $(n \geq m)$ tour exists if and only if the following conditions hold:*

1) n or m is even;

2) $m \notin \{1, 2, 4\}$;

3) $(n, m) \neq (4, 3)$, $(6, 3)$ or $(8, 3)$.

A number of generalisations of this problem have been considered. For example the problem of constructing knight's tours of 2-dimensional chessboards embedded on various surfaces has been considered, such as a sphere [11], a cylinder [55], a torus [54] or the surface of a cube [47]. Further to this one can extend the concept of a knight's tour to higher dimensional chessboards, such as the interior of a cube (or more generally an n-dimensional cuboid). As before we associate with an $n_1 \times n_2 \times \ldots \times n_r$ chessboard a graph, the knight's graph $K(n_1, n_2, \ldots, n_r)$. Here $K(n_1, n_2, \ldots, n_r)$ is the graph G where

$$V(G) = \{(x_1, x_2, \ldots, x_r) : 1 \leq x_i \leq n_i \text{ for all } i \leq r\}$$

and

$$E(G) = \{\big((a_1, a_2, \ldots, a_r), (b_1, b_2, \ldots, b_r)\big) \ : \ \text{there exists } i_1, i_2 \text{ such that}$$
$$|a_{i_1} - b_{i_1}| = 1\,,\ |a_{i_2} - b_{i_2}| = 2 \text{ and } a_i = b_i \text{ for all } i \neq i_1, i_2\}.$$

As before, a closed knight's tour of an $n_1 \times n_2 \times \ldots \times n_r$ chessboard is a Hamiltonian cycle in $K(n_1, n_2, \ldots, n_r)$. Again, we will refer to a closed tour of an $n_1 \times n_2 \times \ldots \times n_r$ chessboard, where the context is clear, as simply an $n_1 \times n_2 \times \ldots \times n_r$ *tour*.

Both Stewart [51] and DeMaio [17] constructed examples of 3-dimensional knight's tours, and in [18] DeMaio and Mathew fully classified the 3-dimensional cuboids on which a chessboard admits a knight's tour.

Theorem 15 (DeMaoi and Mathew [18])**.** *A closed $n \times m \times p$ ($n \geq m \geq p \geq 2$) tour exists if and only if the following conditions hold:*

1) n, m or p is even;

2) $n \geq 4$;

3) $m \geq 3$.

The proof uses a similar method to that of Schwenk [50]. Stewart [51], DeMaio [17] and DeMaio and Mathew [18] all asked whether or not this result could be extended to higher dimensional cuboids. Our main result in this chapter answers this question.

Theorem 16. *Let $r \geq 3$. A closed $n_1 \times n_2 \times \ldots \times n_r$ ($n_1 \geq n_2 \geq \ldots \geq n_r \geq 2$) tour exists if and only if the following conditions hold:*

1) Some n_i is even;

2) $n_1 \geq 4$;

3) $n_2 \geq 3$.

Note that the hypotheses are the same as those in Theorem 15 when $r = 3$. Our proof will use a similar idea to that of Schwenk [50] and DeMaio and Mathew [18]. In the final section of this chapter we will discuss how to apply our methods to the problem of constructing tours using more general knight-like moves.

We will prove Theorem 16 inductively by constructing a small set of tours with specific structural qualities which allow them to be combined to construct larger tours. Specifically it will be necessary to construct examples of tours containing certain patterns on all possible 3-dimensional chessboards. To this end, and for completeness, we will first discuss a proof of Theorem 14, since, by using the ideas in the proof of Theorem 16, this will simplify our construction of the 3-dimensional tours.

3.2 Two dimensional chessboards

In this section we will present a proof of Theorem 14. Our proof follows a similar line to Schwenk's original proof. We will prove the result inductively by constructing a small set of tours with specific structural qualities which allow them to be extended to form larger tours.

Proof of Theorem 14. Firstly we will show that the three conditions are necessary. As with a physical chessboard we can colour the points of our chessboard black and white. We say a point (x, y) is black if $x+y$ is even and white if $x+y$ is odd. It is apparent that each knight's move is between a white point and a black point, that is the graph $K(n, m)$ is bipartite, between the classes of black and white points. Therefore if the number of points in our graph is odd, no Hamiltonian cycle can exist. Indeed suppose we have a cycle $C = \{x_1, x_2, \ldots, x_r\}$ in $K(n, m)$, then the colour of each point in the cycle must alternate. But if r is odd and x_1 is a black point, the so is x_r, and vice versa. In either case x_1 cannot be adjacent to x_r and so C is not a cycle. Therefore, since the number of vertices in $K(n, m)$ is nm, if a Hamiltonian cycle exists at least one of n or m must be even, and so condition 1) is necessary.

If $m = 1$ then $K(n, 1)$ is the empty graph E_n, and so has no Hamiltonian cycle exists. If $m = 2$ then $K(n, 2)$ is not connected, since from $(1, 1)$ we can only reach points of the form $(4k+3, 2)$ or $(4k+1, 1)$, and so no Hamiltonian cycle exists. Finally, if $m = 4$, there is a nice double parity argument, attributed to Louis Pósa in [50], that no Hamiltonian cycle exists. Let us also colour the points (x, y) of the graph $K(n, 4)$ red if $y = 1$ or 4 and blue if $y = 2$ or 3, see Figure 3.2 for an illustration of the case $n = 4$.

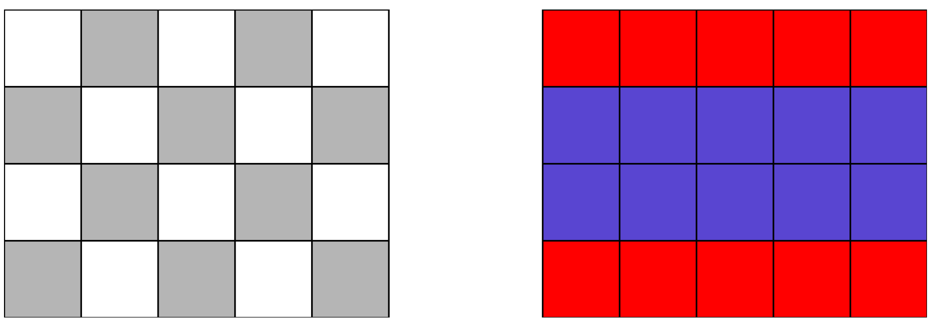

Figure 3.2: The two colourings of a 4×4 chessboard.

As before, every knight's move is between a white and a black point, however we note that from a red point we can only move to a blue point. Therefore if a Hamiltonian cycle exists, it must alternate between red and blue points, since there are an equal number of each. However, as before, it must also alternate between black and white points. Suppose the first point in our cycle is red and white, then the set of red points must be the set of white points, similarly for the other cases. However the four colour classes are clearly distinct, therefore condition 2) is necessary.

In the case of a 6×3 chessboard we see that the cycle in Figure 3.3 must be included, since the points on the far left and far right have degree 2 in $K(6, 3)$, and hence no tour exists.

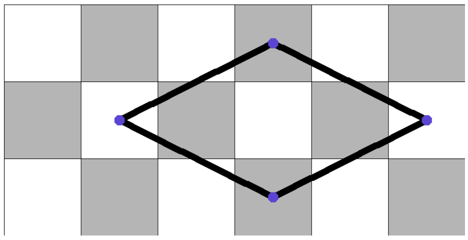

Figure 3.3: A forced cycle in a 6×3 chessboard.

Finally in the case of a 8×3 chessboard we see that the edges in Figure 3.4 must be included and so, by collapsing each path to a point, if a tour exists it must induce

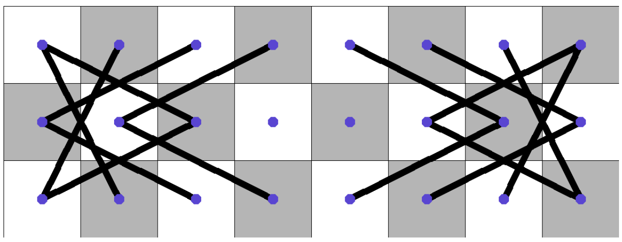

Figure 3.4: Forced edges in a 8×3 chessboard.

a Hamiltonian cycle in the graph in Figure 3.5. However, since all edges adjacent to a

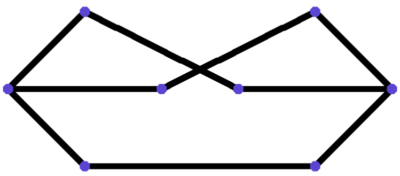

Figure 3.5: A minor of $K(8, 3)$.

vertex of degree 2 must be in the Hamiltonian cycle, then if one were to exist it must in fact be the graph itself, which is not a cycle by inspection. Therefore condition 3) is also necessary.

It remains to show that there exist closed tours on all other sizes of board, which we will do by inducting on a slightly stronger statement. We will require the existence of closed tours with specific structural qualities that will allow us to extend them to larger tours.

We call a tour (open or closed) *seeded* if it includes the edges $\big((1, m-2)\,,\,(2, m)\big)$ and $\big((n-2, 1)\,,\,(n, 2)\big)$, for example the edges between the red vertices in the 10×3 tour in Figure 3.6.

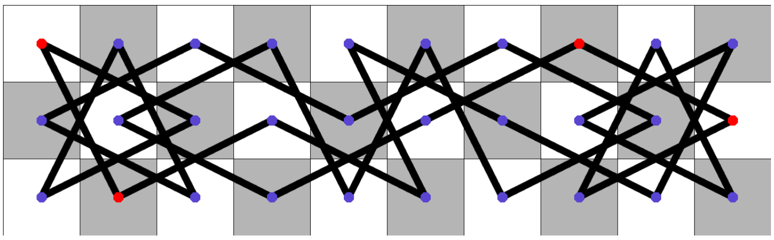

Figure 3.6: A seeded 10×3 tour.

We call an open tour of a $4 \times m$ chessboard a $4 \times m$ *extender* if it is a tour starting at $(4, m-1)$ and ending at $(4, m)$, like the 4×3 extender in Figure 3.7.

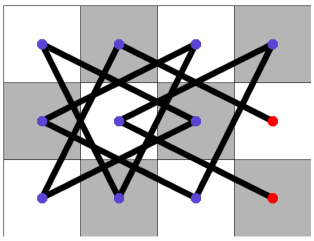

Figure 3.7: A 4×3 extender.

Lemma 17. *For all $m \neq 1, 2$ or 4, there exists a seeded $4 \times m$ extender.*

Proof. Observe that if we place Figure 3.8 below a seeded $4 \times m$ extender and add the edges $((4, m), (2, m+1))$ and $((4, m-1), (3, m+1))$ we form a seeded $4 \times (m+3)$ extender. Therefore if, along with the 4×3 extender of Figure 3.7, we exhibited seeded 4×5 and 4×7 extenders, the result would would follow by induction. An example of seeded 4×5 and 4×7 extenders can be found in Figure 3.9. □

Lemma 18. *For all $m \neq 1, 2$ or 4, if a seeded $n \times m$ tour exists then so does a seeded $(n + 4) \times m$ tour.*

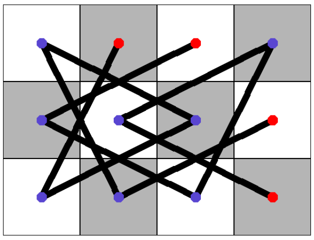

Figure 3.8: Extending an extender

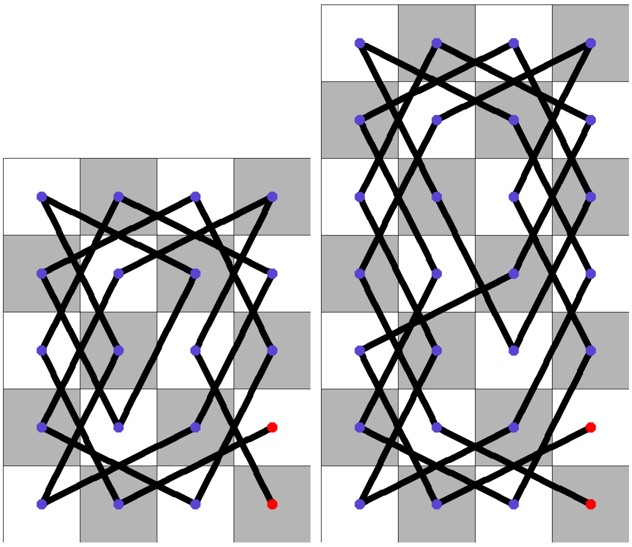

Figure 3.9: Seeded 4×5 and 4×7 extenders.

Proof. Suppose a seeded $n \times m$ tour exists, for some $m \neq 1, 2, 4$. Then by Lemma 17 there exists a seeded $4 \times m$ extender. We place the $4 \times m$ extender to the left of the seeded $n \times m$ tour, as in Figure 3.10. More precisely we can think of this as a subgraph

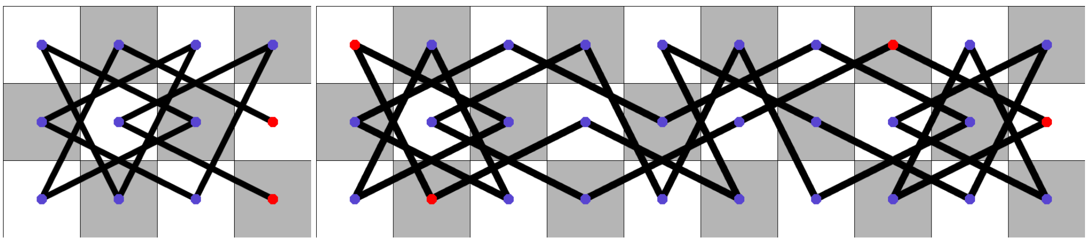

Figure 3.10: Extending a seeded 10×3 tour to a seeded 14×3 tour.

of $K(n+4, m)$, since the induced subgraph of $K(n+4, m)$ on the vertex set $[4] \times [m]$ is just $K(4, m)$, and the induced subgraph on the vertex set $[5, n+4] \times [m]$ is $K(n, m)$. By removing the edge $((6, m), (5, m-2))$ and adding in the two edges $((4, m), (5, m-2))$ and $((4, m-1), (6, m))$ we form a $(n+4) \times m$ tour. Note that, since both the original tour and the extender were seeded, this tour is seeded. $\square$

We note at this point that, since a seeded $n \times m$ tour is equivalently a seeded $m \times n$ tour after a suitable reflection, by induction Lemma 18 implies that if a seeded $n \times m$ tour exists then so does a seeded $(n + 4k) \times (m + 4l)$ tour for all $k, l \in \mathbb{N}$. Therefore in order to prove Theorem 14 it is sufficient to exhibit seeded $n \times m$ tours for all different pairs of residues modulo 4 (excepting the cases where both are odd), and possibly some small cases. A quick check will show it is sufficient to use as base cases seeded 10×3, 12×3, 6×5, 8×5, 6×6, 7×6, 8×6, 8×7 and 8×8 tours, which appear in Figure 3.11.

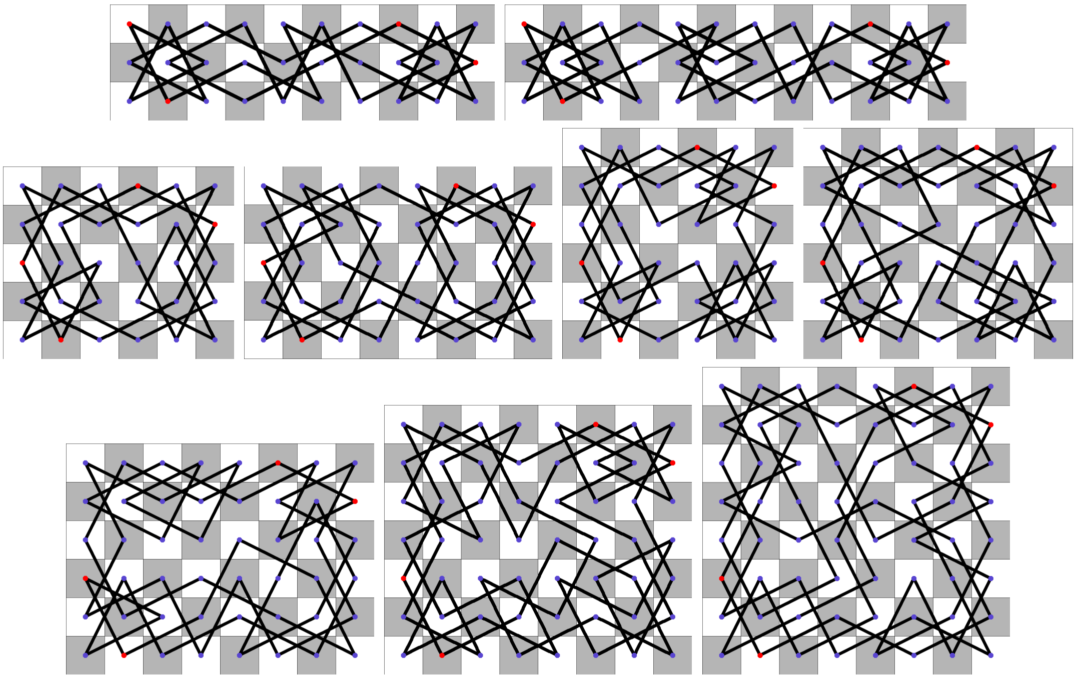

Figure 3.11: Base cases for the induction.

□

3.3 Higher dimensional chessboards

In this section we will prove Theorem 16. Our proof is inductive on the dimension of the chessboard. However a slightly stronger hypothesis is needed to complete the inductive step which will motivate the definition of a site and bi-sited tour which follow.

3.3.1 Using sites to join tours

Given an $n \times m$ tour we call a pair of edges in the tour a *site* if there is a pairing of the endpoints of the edges such that the pairs are each two squares away from each other.

More precisely, we require two edges $((a_1, a_2), (b_1, b_2))$ and $((c_1, c_2), (d_1, d_2))$ such that:

$$(a_1 - c_1, a_2 - c_2) \text{ and } (b_1 - d_1, b_2 - d_2) \in \{(\pm 2, 0), (0, \pm 2)\}, \tag{3.3.1}$$

or

$$(a_1 - d_1, a_2 - d_2) \text{ and } (b_1 - c_1, b_2 - c_2) \in \{(\pm 2, 0), (0, \pm 2)\}. \tag{3.3.2}$$

Examples of the three different type of sites in 2-dimensions are given in Figure 3.12.

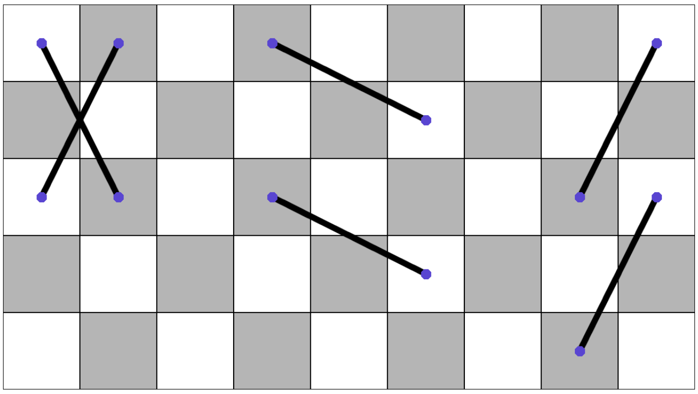

Figure 3.12: Sites in 2-dimensions.

We note that, if some $n \times m$ tour contains a site, then we can use this site to construct an $n \times m \times 2$ tour as follows.

We can think of $K(n, m, 2)$ as containing two disjoint copies of $K(n, m)$, one on top of the other. The first is the subgraph induced on $[n] \times [m] \times \{1\}$ and the second on $[n] \times [m] \times \{2\}$. We can therefore cover the vertices of $K(n, m, 2)$ with two cycles by taking a copy of the $n \times m$ tour in each copy of $K(n, m)$. Suppose the $n \times m$ tour contains a site, that is two edges $((a_1, a_2), (b_1, b_2))$ and $((c_1, c_2), (d_1, d_2))$, without loss of generality satisfying Equation 3.3.1. Equation 3.3.1 guarantees that the points $(a_1, a_2, 1)$ and $(c_1, c_2, 2)$ are a knight's move away, and similarly for $(b_1, b_2, 1)$ and $(d_1, d_2, 2)$. Therefore the edges $((a_1, a_2, 1), (c_1, c_2, 2))$ and $((b_1, b_2, 1), (d_1, d_2, 2))$ are in $K(n, m, 2)$. So if we remove the edge $((a_1, a_2, 1), (b_1, b_2, 1))$ from the first copy of the tour and the edge $((c_1, c_2, 2), (d_1, d_2, 2))$ from the second copy of the tour, and replace them with the edges $((a_1, a_2, 1), (c_1, c_2, 2))$ and $((b_1, b_2, 1), (d_1, d_2, 2))$, we have constructed an $n \times m \times 2$ tour.

For example to construct a $6 \times 5 \times 2$ tour we would first place the left copy of the 6×5 tour in Figure 3.13 on top of the right copy. We would remove the highlighted edges, and add in the edges $((1, 3, 1), (1, 5, 2))$ and $((2, 5, 1), (2, 3, 2))$.

This process can be thought of as in Figure 3.14. We have taken two cycles and removed a single edge from each, the red edges in Figure 3.14, leaving us with two paths.

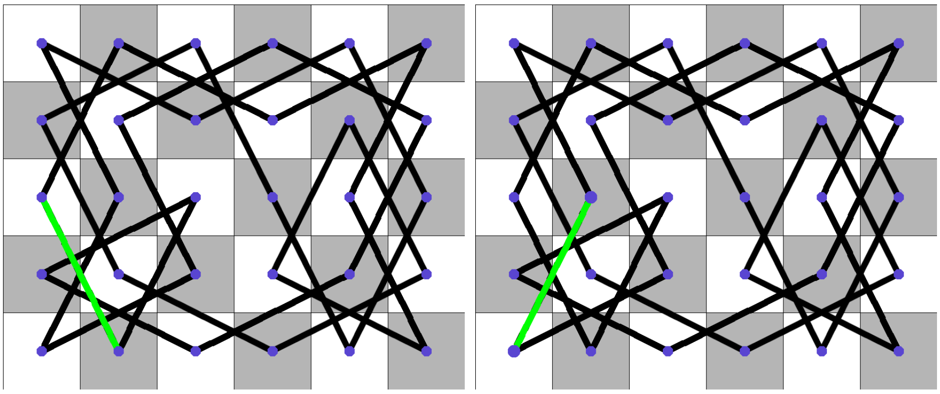

Figure 3.13: Constructing a $6 \times 5 \times 2$ tour.

The definition of a site guarantees us that there is a pairing of the endpoints of these edges such that the two pairs are adjacent. So by adding in the green edges is Figure 3.14, the two options corresponding to the two possible ways to pair the endpoints, we form a larger cycle on the combined vertex set of the original two cycles.

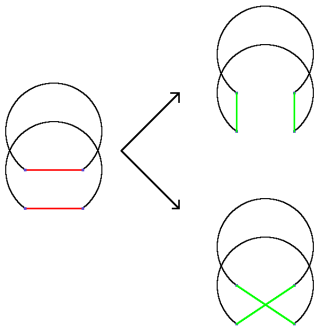

Figure 3.14: Using sites to join cycles.

More generally we say that a pair of edges $\{(\underline{a}, \underline{b}), (\underline{c}, \underline{d})\}$ in $K(n_1, n_2, \ldots, n_r)$ is a *site* if there exists $i_1 \neq i_2$ such that:

$$a_j = c_j \text{ and } b_j = d_j, \text{ for all } j \notin \{i_1, i_2\}$$

and either

$$(a_{i_1} - c_{i_1}, a_{i_2} - c_{i_2}) \text{ and } (b_{i_1} - d_{i_1}, b_{i_2} - d_{i_2}) \in \{(\pm 2, 0), (0, \pm 2)\}, \tag{3.3.3}$$

or

$$(a_{i_1} - d_{i_1}, a_{i_2} - d_{i_2}) \text{ and } (b_{i_1} - c_{i_1}, b_{i_2} - c_{i_2}) \in \{(\pm 2, 0), (0, \pm 2)\}. \tag{3.3.4}$$

We call any knight's tour of a chessboard containing two edge-disjoint sites *bi-sited*.

This is the key idea in the argument, enabling the inductive step to work.

Theorem 19. *For all $k \geq 2$, if a bi-sited $n_1 \times n_2 \times \ldots \times n_r$ tour exists then so does a bi-sited $n_1 \times n_2 \times \ldots \times n_r \times k$ tour.*

Proof. We start by taking k copies of the bi-sited $n_1 \times n_2 \times \ldots \times n_r$ tour and placing them on top of each other, so as to cover the $n_1 \times n_2 \times \ldots \times n_r \times k$ chessboard. We join the first copy to the second copy by the process described in the preceding discussion using the first site on both copies. We then join the second copy to the third copy using the second site, and so on, alternating sites, until we have joined all the copies of the $n_1 \times n_2 \times \ldots \times n_r$ tour together to form an $n_1 \times n_2 \times \ldots \times n_r \times k$ tour. Note that there will be two sites whose edges we have not altered during this process, one in the first copy, and one in the last, and so this tour is also bi-sited. □

Corollary 20. *For all $r \geq 1$ and $k_1, k_2, \ldots, k_r \geq 2$, if an $n \times m$ tour exists then so does an $n \times m \times k_1 \times k_2 \times \ldots \times k_r$ tour.*

Proof. We note that in $K(n, m)$ the vertex $(1, 1)$ has degree 2, and so any Hamiltonian cycle must contain both edges adjacent to it, that is the edges $((1, 1), (3, 2))$ and $((1, 1), (2, 3))$.

Similarly of the (at most) 4 edges adjacent to the point $(1, 3)$ at least 2 of them must be included in the tour, but 3 of them form sites with the 2 forced edges, as in Figure 3.15.

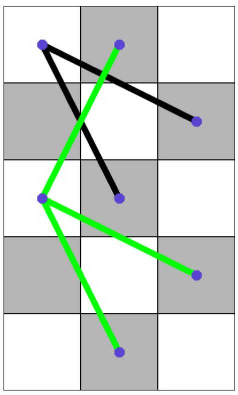

Figure 3.15: Forced edges in the corner of a chessboard.

By a similar argument a site exists in each corner of the board, and since $n \geq 6$ (see the statement of Theorem 14) at least two of these sites are edge-disjoint. The result then follows by repeated applications of Theorem 19. □

As an example see Figure 3.16 for an illustration of a bi-sited 10×3 tour, where the sites are the highlighted edges.

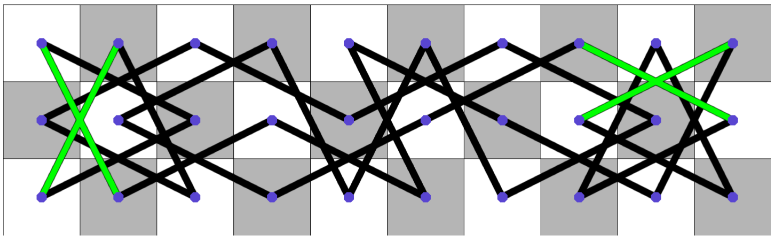

Figure 3.16: A bi-sited 10×3 tour.

So already by combining Theorem 14 and Corollary 20 we have shown the existence of a large class of higher dimensional tours. We aim to classify all tourable chessboards by constructing bi-sited examples in small dimensions. In particular if we could construct bi-sited examples of all the tours in Theorem 15 it would be sufficient to prove Theorem 16. We defer the proof of the following Lemma to the next subsection:

Lemma 21. *A bi-sited $n \times m \times p$ $(n \geq m \geq p)$ tour exists if and only if the following conditions hold:*

1) n, m or p is even;

2) $n \geq 4$;

3) $m \geq 3$.

Remark 22. *Given $n_1, \ldots n_r$ and ψ a permutation on $[r]$ it is clear that $K(n_1, \ldots, n_r)$ is isomorphic to $K(n_{\psi(1)}, \ldots, n_{\psi(r)})$ and so $K(n_1, \ldots, n_r)$ contains a Hamiltonian cycle if and only if $K(n_{\psi(1)}, \ldots, n_{\psi(r)})$ does.*

Proof of Theorem 16. As in the proof of Theorem 14 we can see that condition 1) is necessary due to a simple parity consideration. If either condition 2) or condition 3) does not hold then it is a simple check that $K(n_1, n_2, \ldots, n_r)$ is disconnected.

Given an $n_1 \times n_2 \times \ldots \times n_r$ chessboard (with $n_1 \geq n_2 \geq \ldots \geq n_r$) such that some n_i is even, then, unless $n_i = 2$ for all $i \geq 2$ or $n_i \leq 3$ for all i, there is some triple $n_{i_1}, n_{i_2}, n_{i_3}$ that satisfies the conditions of Lemma 21. Let the remaining n_js be $\{m_4, m_5, \ldots, m_r\}$, in any order. By Lemma 21 there exists a bi-sited $n_{i_1} \times n_{i_2} \times n_{i_3}$ tour, and so by repeated applications of Theorem 19 there exists a bi-sited $n_{i_1} \times n_{i_2} \times n_{i_3} \times m_4 \times \ldots \times m_r$ tour. Therefore, by Remark 22, there exists an $n_1 \times n_2 \times \ldots \times n_r$ tour. $\square$

We note that an immediate consequence of Theorem 16 is:

Corollary 23. *For all $r \geq 3$, $K(n_1, n_2, \ldots, n_r)$ is Hamiltonian if and only if some n_i is even and $K(n_1, n_2, \ldots, n_r)$ is connected.*

3.3.2 Three dimensional chessboards

In this subsection we will present a proof of Lemma 21. As in Section 3.2 the proof will follow along similar lines to DeMaio and Mathew's [18] original argument. However combining the methods of Subsection 3.3.1 with the constructions in Section 3.2 will allow us to significantly shorten the presentation.

By Corollary 20 we can construct bi-sited $n \times m \times p$ tours whenever an $n \times m$ tour exists. Also, by Remark 22 if such a tour exists for $n \times m \times p$ then it also does for all permutations of n, m and p. We will split the remaining tours into cases:

- $n \times m \times 2$ and $n \times m \times 4$ tours for $n, m \geq 5$;
- $4 \times 4 \times p$ tours for $p \geq 2$;
- $4 \times 3 \times p$ tours for odd $p \geq 3$ and $p = 2, 4, 6, 8$;
- $4 \times 2 \times p$ tours for $p \geq 3$;
- $3 \times 2 \times p$ tours for odd $p \geq 5$ and $p = 4, 6, 8$;
- a $3 \times 3 \times 6$ tour and a $3 \times 3 \times 8$ tour.

Note that some of these cases will overlap in a small number of tours. Let us first consider $n \times m \times 2$ and $n \times m \times 4$ chessboards for $n, m \geq 5$ and odd.

Note that given an open tour of an $n \times m$ chessboard which starts at (n, m) and ends two squares above at $(n, m-2)$ we can construct a closed tour of the $n \times m \times 2$ chessboard by putting two copies of the open tour above one another and adding in the lines $((n, m, 1), (n, m-2, 2))$ and $((n, m-2, 1), (n, m, 2))$. We illustrate this with an example of such an open tour on a 5×5 chessboard in Figure 3.17.

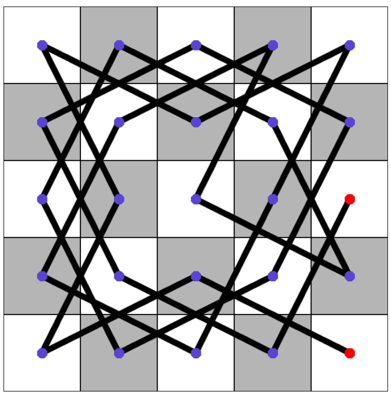

Figure 3.17: An open 5×5 tour.

Also, since this tour is seeded, using the $4 \times m$ extenders of Section 3.2 we could construct such open tours, and so construct closed $n \times m \times 2$ tours, for all $n, m \equiv 1 \pmod 4$. By a similar argument, using the open 7×5, 5×7 and 7×7 tours in Figure 3.18, we can construct closed $n \times m \times 2$ tours for all $n, m \geq 5$ and odd. Note that these tours are bi-sited, having for example a site in the bottom left corner of both layers. Finally, since these two sites are directly on top of each other, we can use the same method as in Theorem 19 to construct bi-sited $n \times m \times 2k$ tours for all $n, m \geq 5$ and odd, in particular for $n \times m \times 4$ chessboards.

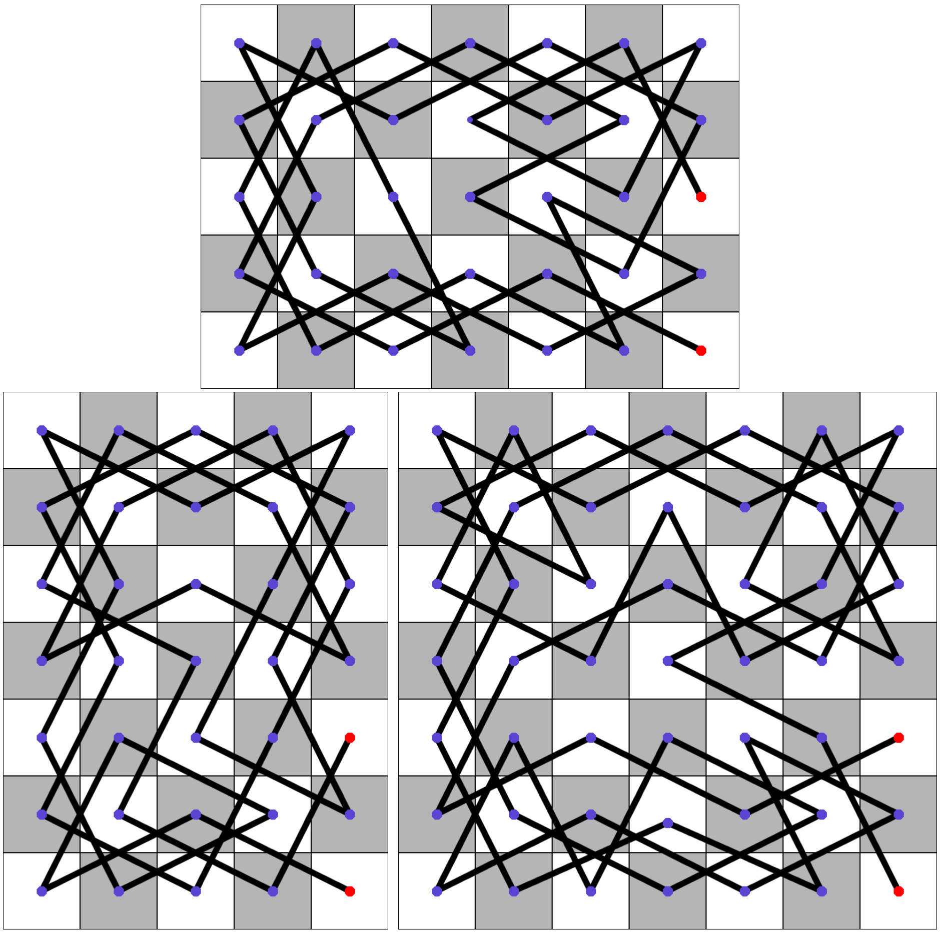

Figure 3.18: Open 7×5, 5×7 and 7×7 tours.

So, by the preceding discussion and Theorem 14, we have constructed $n \times m \times p$ tours for all possible triples n, m, p, with $n, m \geq 5$. So all the remaining cases have at least two sides smaller than 5. Let us first consider tours with a side of length 4.

The method we have used so far to draw tours will be insufficient to demonstrate more complicated 3-dimensional tours so we will simply present them layer by layer with each square numbered with the order it appears in the tour, starting with the topmost layer to the left and so on. Sites will be indicated by numbers coloured red.

Firstly we will construct bi-sited $4 \times 4 \times p$ tours for all $p \geq 2$. In Figure 3.19 we exhibit a $4 \times 4 \times 2$ and $4 \times 4 \times 3$ tour. Notice the sites in the top left corners of the top and

1	30	13	26
14	27	2	31
29	32	25	12
24	15	28	3

18	21	6	11
7	10	19	22
20	17	8	5
9	4	23	16

1	46	19	34
20	33	2	47
45	48	35	22
36	21	32	3

44	11	6	23
5	8	41	14
12	43	10	7
9	4	13	42

29	24	39	18
40	15	30	27
25	28	17	38
16	37	26	31

Figure 3.19: A $4 \times 4 \times 2$ and $4 \times 4 \times 3$ tour.

bottom layers of each of them, that is the lines $1-32$, $29-30$, $17-18$ and $20-21$ in the $4 \times 4 \times 2$ tour and the lines $1-48$, $45-46$, $28-29$ and $24-25$ in the $4 \times 4 \times 3$ tour . As before we can use the methods of Theorem 19 to stack any number of these on top of each other and construct bi-sited $4 \times 4 \times p$ tours for all p.

More concretely we can form a $4 \times 4 \times 4$ tour by removing the line $20-21$ from a copy of a $4 \times 4 \times 2$ tour and placing it on top of another copy with the line $1'-32'$ removed, then add in the lines $20-1'$ and $21-32'$. In a similar fashion we can add any number of $4 \times 4 \times 2$ and $4 \times 4 \times 3$ tours together.

Next we will construct bi-sited $4 \times 3 \times p$ tours for all odd $p \geq 3$ and $p = 2, 4, 6, 8$. In Figure 3.20 we exhibit a $3 \times 4 \times 2$ and a $3 \times 4 \times 3$ tour with sites in the top left corners of the top and bottom layers.

By the same method as above we can use these to construct bi-sited $3 \times 4 \times p$ tours for all $p \geq 2$.

Next we will construct bisited $4 \times 2 \times p$ tours for all $p \geq 3$. A $4 \times 2 \times 2$ tour does not exist, so to proceed along similar lines we will have to use as our base cases a $4 \times 2 \times 3$, a $4 \times 2 \times 4$ and a $4 \times 2 \times 5$ tour.

By Remark 22 we can use for our $4 \times 2 \times 3$ tour the $3 \times 4 \times 2$ tour in Figure 3.20. We can construct a $6 \times 4 \times 2$ tour by placing two copies of the $3 \times 4 \times 2$ tour side by side, removing the $11-12$ line from the left copy and the $1'-2'$ line from the right copy and

1	14	3
4	11	24
13	2	15
16	23	12

6	19	8
9	22	5
18	7	20
21	10	17

1	28	3
4	25	36
27	2	23
24	35	26

10	17	8
7	20	5
22	9	18
19	6	21

15	32	11
12	29	14
33	16	31
30	13	34

Figure 3.20: A $3 \times 4 \times 2$ and $3 \times 4 \times 3$ tour.

adding in the $11 - 1'$ and $12 - 2'$ lines. We can continue adding tours in this way way to construct bi-sited $p \times 4 \times 2$ tours for all $p \equiv 0 \pmod 3$.

Similarly we can add the $4 \times 4 \times 2$ tour to the left of the $3 \times 4 \times 2$ tour we constructed by removing the $2 - 3$ line from the $4 \times 4 \times 2$ tour and the $1' - 2'$ from the $4 \times 3 \times 2$ tour and adding in the $2 - 1'$ and $3 - 2'$ lines. We can do the same thing to any of the $p \times 4 \times 2$ we constructed in the preceding paragraph to construct bi-sited $p \times 4 \times 2$ tours for all $p \equiv 1 \pmod 3$ and $p \geq 3$.

Finally, if we are able to construct a bi-sited $5 \times 4 \times 2$ tour that includes the line $\big((4,2,1)\,,\,(5,4,1)\big)$, we can do the same for the case $p \equiv 2 \pmod 3$ and $p \geq 3$. See Figure 3.21 for an example of such a tour, it is a simple check that the tours constructed in this way are bi-sited.

1	22	5	26	9
6	27	40	21	4
23	2	29	8	25
28	7	24	3	20

16	31	10	37	14
11	36	15	32	19
30	17	34	13	38
35	12	39	18	33

Figure 3.21: A $5 \times 4 \times 2$ tour.

Next we will construct bi-sited $3 \times 2 \times p$ for odd $p \geq 5$ and $p = 4, 6, 8$. Using the $3 \times 4 \times 2$ tour in Figure 3.20 we can construct a $3 \times 8 \times 2$ tour by placing two copies of tour end on end, removing the line $15 - 16$ in the copy above and the line $8' - 9'$ in the copy below and adding in the lines $15 - 8'$ and $16 - 9'$. We can continue adding tours in this way to construct $3 \times p \times 2$ tours for all $p \equiv 0 \text{ mod}(4)$, and in particular $p = 4, 8$.

So to complete this case it will be sufficient to exhibit tours of size $3 \times 5 \times 2$, $3 \times 6 \times 2$ and $3 \times 7 \times 2$ which include the line $((3,1,2),(1,2,2))$, which appear in Figure 3.22.

1	14	5
4	27	2
15	6	17
26	3	20
19	16	25

12	9	30
29	22	11
10	13	8
21	28	23
24	7	18

1	4	29
26	15	36
7	30	5
14	21	16
17	6	9
10	13	20

28	31	2
35	24	27
32	3	34
25	12	23
8	33	18
19	22	11

1	40	11
12	5	42
41	34	39
4	17	6
33	26	35
28	7	18
19	32	27

10	37	2
3	14	9
38	23	36
13	8	15
24	31	22
21	16	29
30	25	20

Figure 3.22: A $3 \times 5 \times 2$, $3 \times 6 \times 2$ and $3 \times 7 \times 2$ tour.

The only remaining cases are that of a $3 \times 3 \times 6$ and a $3 \times 3 \times 8$ tour. Firstly if we look back at the $3 \times 4 \times 3$ tour we can join two of these together to form a $3 \times 8 \times 3$ tour by deleting the $23 - 24$ line in the copy above and the $7' - 8'$ line in the copy below and adding in the lines $24 - 7'$ and $23 - 8'$. Finally, a bi-sited $3 \times 6 \times 3$ tour appears in Figure 3.23.

1	4	53
52	21	2
3	54	5
22	51	24
49	6	43
44	23	50

8	11	32
33	26	9
10	7	12
25	38	27
46	13	40
39	28	45

31	20	15
16	35	30
19	14	17
34	29	36
41	18	47
48	37	42

Figure 3.23: A $3 \times 6 \times 3$ tour.

3.4 Generalised knight's tours

The knight's tour is a specific case of many general questions. A natural one to ask would be, what about more general moves? For example instead of the knight being able to move $(\pm 1, \pm 2)$ or $(\pm 2, \pm 1)$ what if the knight could move $(\pm\alpha, \pm\beta)$ or $(\pm\beta, \pm\alpha)$, for some other $\alpha, \beta \in \mathbb{N}$?

We define an (α, β)-*tour* of an $n_1 \times n_2 \times \ldots \times n_r$ chessboard to be a closed tour of the board only using moves of the form $(0, 0, \ldots, 0, \pm\alpha, 0, \ldots, 0, \pm\beta, 0, \ldots, 0)$ or of the form $(0, 0, \ldots, 0, \pm\beta, 0, \ldots, 0, \pm\alpha, 0, \ldots, 0)$, and $K_{\alpha,\beta}(n, m)$ in the obvious way. We refer to Figure 3.24 for an example where $\alpha = 2$ and $\beta = 3$.

Tours of this kind have been considered by various authors, an early example being the construction by Frost [24] of $(1, 4)$ and $(2, 3)$-tours of the 10×10 chessboard. Dawson [16] constructed open $(1, 2k)$-tours of $(2k + 1) \times 4k$ chessboard. Jellis [33] called a chess piece moving in this fashion a 'leaper', and established some properties of the graphs $K_{\alpha,\beta}(n, m)$ for general α and β. Knuth [37] showed that if $\alpha + \beta$ and $\alpha - \beta$ are relatively prime then $K_{\alpha,\beta}(n, m)$ is connected for $n \geq 2\beta$ and $m \geq \alpha + \beta$, otherwise $K_{a,b}(n, m)$ is disconnected. He also showed that in the specific case of $K_{\alpha,\alpha+1}(n, m)$ the smallest choice of n and m (in terms of their product) for which $K_{\alpha,\alpha+1}(n, m)$ has a Hamiltonian cycle is $K_{\alpha,\alpha+1}(4\alpha + 2, 4\alpha + 2)$. Willcocks [34] conjectured that there exists an (α, β)-tour of an $(2\alpha+2\beta)\times(2\alpha+2\beta)$ chessboard whenever $\alpha+\beta$ and $\alpha-\beta$ are relatively prime. Knuth [37] showed this is true for the case $\alpha = 1$, constructing a $(1, 2\beta)$-tour of a $(4\beta + 2) \times (4\beta + 2)$ chessboard for all β. Knuth also constructed a a $(1, 2\beta)$-tour of a $(4\beta + 1) \times (4\beta + 2)$ chessboard, and showed that this is the board of smallest area that is tourable in this case. The techniques of Section 3.3 are also applicable to generalised knight's tours.

We define an α-*site* to be a pair of lines in an (α, β)-tour such that there is a pairing of the endpoints of the edges such that the pairs are each α squares away from each other in

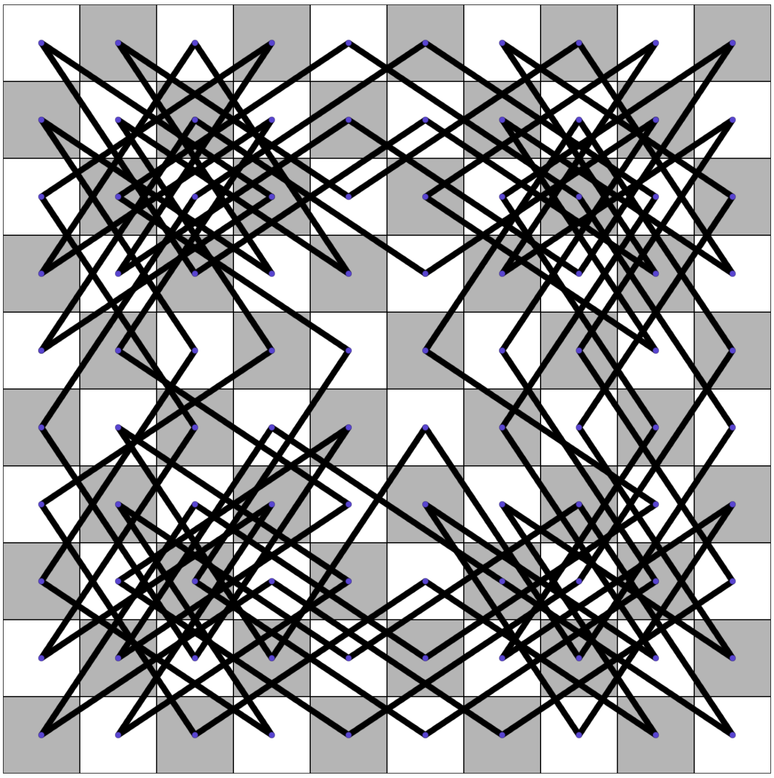

Figure 3.24: A 10×10 $(2, 3)$-tour constructed by A. H. Frost [24].

some direction, and a *β-site* in the same way. To be explicit we say that a pair of edges $\{(\underline{a}, \underline{b}), (\underline{c}, \underline{d})\}$ in $K(n_1, n_2, \ldots, n_r)$ is an *α-site* if there exists $i_1 \neq i_2$ such that:

$$a_j = c_j \text{ and } b_j = d_j, \text{ for all } j \notin \{i_1, i_2\}$$

and either

$$(a_{i_1} - c_{i_1}, a_{i_2} - c_{i_2}) \text{ and } (b_{i_1} - d_{i_1}, b_{i_2} - d_{i_2}) \in \{(\pm\alpha, 0), (0, \pm\alpha)\}, \tag{3.4.1}$$

or

$$(a_{i_1} - d_{i_1}, a_{i_2} - d_{i_2}) \text{ and } (b_{i_1} - c_{i_1}, b_{i_2} - c_{i_2}) \in \{(\pm\alpha, 0), (0, \pm\alpha)\}. \tag{3.4.2}$$

We call a (α, β)-tour that contains a set of 2 α-sites and 2 β-sites, all edge-disjoint, *(α, β)-sited*. We can prove an analogue of Theorem 19 for generalised knight's tours.

Theorem 24. *For all $k \geq \alpha + \beta + 1$, if an (α, β)-sited $n_1 \times n_2 \times \ldots \times n_r$ (α, β)-tour exists then so does an (α, β)-sited $n_1 \times n_2 \times \ldots \times n_r \times k$ (α, β)-tour.*

Proof. As in the proof of Theorem 19 we start by covering the $n_1 \times n_2 \times \ldots \times n_r \times k$ chessboard with k copies of the (α, β)-sited $n_1 \times n_2 \times \ldots \times n_r$ (α, β)-tour. In the case of a $(1,2)$ knight, we could use a 2-site to join together two tours which were 1 layer apart. By a similar argument we can use an α-site to join together two tours which are β layers apart, and similarly for a β-site.

Indeed suppose we have some α-site in the $n_1 \times n_2 \times \ldots \times n_r$ tour, without loss of generality a pair of edges $(\underline{a}, \underline{b})$ and $(\underline{c}, \underline{d})$ that satisfy Equation 3.4.1. Equation 3.4.1 guarantees that the vertices $(\underline{a}, 1)$ and $(\underline{c}, \beta+1)$ are adjacent in $K_{\alpha,\beta}(n_1, n_2, \ldots, n_r, k)$, as are $(\underline{b}, 1)$ and $(\underline{d}, \beta+1)$. Therefore we can remove the edge $((\underline{a}, 1), (\underline{b}, 1))$ from the tour on the layer $n_1 \times n_2 \times \ldots \times n_r \times \{1\}$ and the edge $((\underline{c}, \beta+1), (\underline{d}, \beta+1))$ from the tour on the layer $n_1 \times n_2 \times \ldots \times n_r \times \{\beta+1\}$ and add in the edges $((\underline{a}, 1), (\underline{c}, \beta+1))$ and $((\underline{b}, 1), (\underline{d}, \beta+1))$. The forms a tour of the two layers of the chessboard $n_1 \times n_2 \times \ldots \times n_r \times \{1\}$ and $n_1 \times n_2 \times \ldots \times n_r \times \{\beta+1\}$, which still contains all the other α and β-sites in the original $n_1 \times n_2 \times \ldots \times n_r$ tours.

In the case where $\beta = 1$ we could then follow the proof of Theorem 19. We would use the α-sites to join each tour to the tour on the next layer one by one, alternating the α-site we use each time, until we had joined all the tours into a single $n_1 \times n_2 \times \ldots \times n_r \times k$ tour.

The situation for general α and β is similar, except we can only join tours that are α or β layers apart. So instead of just going through the layers one by one we would need to find a path going through all the layers, which only ever moves exactly α or β layers at a time. In fact we can get by with slightly less.

We can use the α-sites to join the first layer to the $(\beta+1)^{\text{th}}$ layer, then the $(\beta+1)^{\text{th}}$ layer to the $(2\beta+1)^{\text{th}}$ layer, and so on, alternating the α-site we use on each level. Eventually we have formed a tour of the set of layers $n_1 \times n_2 \times \ldots \times n_r \times \{p\}$ such that $p \equiv 1 \pmod{\beta}$. We do this for each equivalence class of layers (mod β). At this point we have a set of β cycles $\{C_1, C_2, \ldots, C_\beta\}$, each one touring an equivalence class of layers (mod β). We now use the α-sites to join these tours together.

Since for a tour to exist $\alpha+\beta$ and $\alpha-\beta$ must be coprime, in particular α and β must be coprime. We can use the β-sites to join the first layer to the $(\alpha+1)^{\text{th}}$ layer then the $(\alpha+1)^{\text{th}}$ layer to the $(2\alpha+1)^{\text{th}}$ layer, and so on until the $(\beta\alpha+1)^{\text{th}}$ layer alternating the β-site we use on each level. If we let i_j be the residue of $j\alpha+1 \pmod{\beta}$ we can think of this process as joining C_{i_1} to C_{i_2}, then C_{i_2} to C_{i_3}, and so on. Since α and β are coprime each of these layers lies in a different equivalence class (mod β), that is $\{i_1, i_2, \ldots, i_\beta\} = [\beta]$, and so at the end of this process we have just a single tour on all the layers, that is an $n_1 \times n_2 \times \ldots \times n_r \times k$ (α, β)-tour.

In the proceeding paragraph it was necessary that $k \geq \beta\alpha + 1$. However if instead, when joining C_{i_j} to $C_{i_{j+1}}$, we used a β-site not in the $(j\alpha+1)^{\text{th}}$ layer, but in the $i_j{}^{\text{th}}$ layer, we could construct an $n_1 \times n_2 \times \ldots \times n_r \times k$ (α, β)-tour as long as $k \geq \alpha + \beta + 1$.

It is a simple check that the tour constructed in this way is (α, β)-sited, for example having an unused α-site and β-site on the 1^{st} and k^{th} layer. □

We note that by a similar argument to Corollary 20, if an $n \times m$ (α, β)-tour exists, for n, m sufficiently large, then it must be (α, β)-sited. For example, suppose $\alpha > \beta$. Note that, since a tour exists, $\alpha \neq \beta$. The vertex $(1, 1)$ has degree 2 in $K_{\alpha,\beta}(n, m)$. So any Hamiltonian cycle must contain both edges adjacent to it, that is the edges $\big((1, 1), (\alpha + 1, \beta + 1)\big)$ and $\big((1, 1), (\beta + 1, \alpha + 1)\big)$. Furthermore of the 3 edges adjacent to the vertex $(1, \beta+1)$ (which end at $(\alpha+1, 1)$, $(\alpha+1, 2\beta+1)$, $(\beta+1, \beta+\alpha+1)$) at least 2 must be in the tour, but 2 of them form β-sites with the two forced edges. Similarly of the 4 edges adjacent to $(1, \alpha+1)$, 3 of them form α-site with the two forced edges. So, as long as the chessboard is sufficiently large to ensure that the sites in each corner are edge-disjoint, any tour must contain at least 2 α-sites and 2 β-sites, all edge-disjoint. In particular if $n, m \geq 2\alpha + 2\beta + 1$ this will be the case.

Corollary 25. *For all $r \geq 1$ and $k_1, k_2, \ldots, k_r \geq \alpha+\beta+1$, if an $n\times m$ (α, β)-tour exists, for $n, m \geq 2\alpha + 2\beta + 1$, then so does an $n \times m \times k_1 \times k_2 \times \ldots \times k_r$ (α, β)-tour.*

It is not known in general for which α, β (α, β)-tours exist on sufficiently large chessboards, with at least one side of even length. Corollary 25 reduces the multi-dimensional problem to the 2-dimensional case. Since in 2-dimensional boards there is a lot of structure that is guaranteed around the edges of the board in any tour, it does not seem an infeasible task, for any specific $\alpha, \beta \in \mathbb{N}$ with $\alpha+\beta$ and $\alpha-\beta$ coprime, to prove that sufficiently large tours exist by a similar method to Section 3.2. However for large values of α and β, constructing the base cases for an inductive proof will become computationally infeasible, for example even in the case of $(2, 3)$-tours the board with the smallest area on which a closed tour exists is the 15×8 chessboard [32]. Very recently Kamčev [35] showed that in the case $\alpha = 1$ closed $(1, 2\beta)$-tours exist on $n \times n$ chessboards for all sufficiently large, even n and all β. Kamčev also proved the same result in the specific cases of $(2, 3)$ and $(2, 5)$-tours.

Conjecture 26. *Let α and β be such that $\alpha+\beta$ and $\alpha-\beta$ are relatively prime. Then for all sufficiently large n and m with at least one of n or m even there exists an (α, β)-tour of an $n \times m$ chessboard.*

It is natural to consider even more general knight-like moves. For example given $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{N}$ we can consider $(\alpha_1, \alpha_2, \ldots, \alpha_s)$-tours on $n_1 \times n_2 ... \times n_r$ chessboards as long as $r \geq s$. It is not hard to show that given $\alpha_1, \alpha_2, \ldots, \alpha_s$ and $r > s$, $K_{\alpha_1,\alpha_2,\ldots,\alpha_s}(n_1\ldots,n_r)$ is connected, for sufficiently large n_i, if and only if $\sum_i \alpha_i \equiv 1 \bmod(2)$ and the greatest common factor of $\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ is 1. If $r = s$ then we require the additional constraint that at least one of the α_i is even.

Conjecture 27. *Let* r,s *and* $\alpha_1, \alpha_2, \ldots, \alpha_s$ *satisfy the conditions above. Then for all sufficiently large* $n_1, n_2, \ldots, n_r$ *with some* n_i *even there exists an* $(\alpha_1, \alpha_2, \ldots, \alpha_s)$*-tour of an* $n_1 \times n_2 \times \ldots \times n_r$ *chessboard.*

The results in this chapter were submitted for publication in the Electronic Journal of Combinatorics in February 2012. A proof of the main result was also submitted to the same journal at around the same time by Bruno Golénia and Sylvain Golénia. On the advice of the editors we merged the papers and it was published in [21].

Chapter 4

An n-in-a-row game

4.1 Introduction

A *positional game* is a pair $(X, \mathcal{F})$ where X is a set and $\mathcal{F} \subset \mathbb{P}(X)$. We call X the *board*, and the members $F \in \mathcal{F}$ are *winning sets*. We call a positional game *finite* if X is finite. The game is played by two players, Red and Blue, who alternately claim unclaimed points from the board. Given a particular play of that game, that is a sequences of moves $(r_1, b_1, r_2, b_2 \ldots)$, the winner is the first player to claim all points from a winning set. If at no point during the game either player achieves this, the game is a draw. We call a game, $(X, \mathcal{F})$, a *first player win* if the first player has a winning strategy, and similarly a *second player win* if the second player has a winning strategy. If both players have a drawing strategy then we call the game a *draw*. In the case where X is finite it is a straightforward application of De Morgan's laws that one of these three cases must hold. Indeed if we think of a particular play of a game as being a sequence of moves $\{r_1, b_2, r_3, b_4 \ldots\}$ which eventually cover the whole board, then the outcome is determined by this sequence. So if the first player has a winning strategy then

$$\exists r_1 \forall b_2 \exists r_3 \forall b_2 \ldots \text{ such that } \{r_1, b_2, r_3, b_4 \ldots\} \text{ is a first player win}$$

and if the second player has a winning strategy then

$$\forall r_1 \exists b_2 \forall r_3 \exists b_2 \ldots \text{ such that } \{r_1, b_2, r_3, b_4 \ldots\} \text{ is a second player win.}$$

Therefore if neither happens, then by De Morgan's laws both

$$\forall r_1 \exists b_2 \forall r_3 \exists b_2 \ldots \text{ such that } \{r_1, b_2, r_3, b_4 \ldots\} \text{ is not a first player win,}$$

and

$$\exists r_1 \forall b_2 \exists r_3 \forall b_2 \ldots \text{ such that } \{r_1, b_2, r_3, b_4 \ldots\} \text{ is not a second player win}$$

and so both players have a drawing strategy. It is a folklore theorem that in fact any finite positional game is either a first player win, or a draw. The following argument is usually referred to as strategy stealing.

Theorem 28. *[Folklore] Let $(X, \mathcal{F})$ be a finite positional game, then $(X, \mathcal{F})$ is either a first player win or a draw.*

Proof. Let us assume that Blue has a winning strategy, Φ. We describe a winning strategy for Red as follows. Red claims his first point arbitrarily, and from this point onwards in the game he ignores that point and pretends to be Blue. That is he responds to each of Blue's moves according to the strategy Φ, as if he had not taken the first point and Blue is the first player. If at any point the strategy calls for him to claim a point that he has already taken but ignored, he simply claims another point arbitrarily and ignores that one. Since the arbitrary extra point can only help Red, and since Φ was a winning strategy, Red wins the game, however this contradicts the assumption that Blue had a winning strategy . □

Theorem 28 also hold in the case where X is infinite, if all the winning sets are finite. We call such a game a *semi-infinite* positional game. The *n-in-a-row game* is a semi-infinite positional game played on $\mathbb{Z}^2$ where the winning sets are any n consecutive points in a row, either horizontally, vertically or diagonally (that is, at 45°). By the above, for any particular n, the n-in-a-row game is either a first player win or a draw. For $n \leq 4$ it is possible by case checking to show that the n-in-a-row game is a first player win. For $n \geq 8$ it has been shown that the n-in-a-row game is a draw, the best known bounds come from pairing strategies. A *pairing strategy* starts with a set of disjoint pairs of points from the board, $\bigcup_i P_i$, $P_i = \{p_i, q_i\}$, such that every winning set contains some pair. Blue's strategy is then, whenever Red picks a point p_i, to pick the corresponding q_i, and vice versa. It is clear that at the end of the game Blue has claimed a transversal of the pairs, and so Red has not fully claimed any winning sets.

For example Berlekamp, Conway, and Guy [9] used the pairing strategy in Figure 4.1 to show that the 8-in-a-row game played on a torus is a draw. Breaker's strategy is to reply to each of Maker's moves by looking in the line in the direction indicated by the square Maker just played in, and claiming the nearest point in that line with the same direction marked. It is a simple check that this is a pairing strategy.

By extending the pattern to $\mathbb{Z}^2$ in the obvious way, one can show the 9-in-a-row game

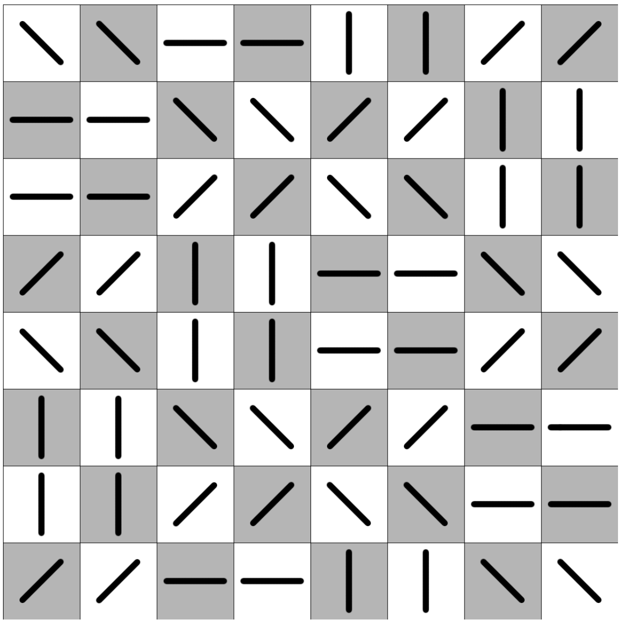

Figure 4.1: A pairing strategy on the 8×8 torus.

is a draw, a result previously shown by Pollak and Shannon. By a similar method Zetters [56], answering a question of Guy and Selfridge, showed the 8-in-a-row game was a draw. It is believed that for $n = 5$ the game is a first player win, and a draw for $n \geq 6$.

Question 29. *Is the n-in-a-row game a first player win or a draw for $n = 5, 6, 7$?*

In this chapter we consider a related game. Given a function $f : \mathbb{N} \to \mathbb{N}$ we define the (n, f) *game* to be a positional game played on the same board with the same winning sets as n-in-a-row, however now at time t a player claims $f(t)$ points. By this we mean that in the first turn Red will claim $f(1)$ points, and in the second turn Blue will claim $f(2)$ points, and so on. The n-in-a-row game corresponds to the $(n, 1)$ game, where 1 is the constant function taking value 1. Note that, for general f, this is not strictly a positional game, since the number of points claimed on each turn changes. In this note we will consider the (n, ι) game, where ι is the identity function. So for example, in the (n, ι) game, in the first turn Red will claim 1 point, and in the second turn Blue will claim 2 points, and so on. Unlike the n-in-a-row game the (n, ι) game is never (with perfect play) a draw, since at time n some player will claim n points and so can claim a winning set. A small case analysis shows that player 1 wins for $n = 1, 3, 4, 6, 7$ and player 2 wins for $n = 2, 5$, and so a strategy stealing argument cannot apply. We note however that for a large class of f the (n, f) game can be shown to be a first player win by a strategy stealing argument.

Proposition 30. *Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f(2t-1) \geq f(2t)$ and $f(2t+1) \geq f(2t)$ for* *all* $t \geq 1$. *Then the* (n, f) *game is a first player win.*

Proof. Suppose that Blue has a winning strategy, Φ, as before we let Red claim $f(1)$ points arbitrarily, and from that point onwards in the game he ignores those points and pretends to be Blue. That is he responds to each of Blue's moves according to the strategy Φ, as if he had not taken the first $f(1)$ points and Blue is the first player. If at any point the strategy calls for him to claim a point that he has already taken but ignored, he simply claims another point arbitrarily and ignores that one.

Since $f(2t+1) \geq f(2t)$ Red always claims at least enough points in his turns to follow Blue's strategy, and if he is required to take extra points he can claim them arbitrarily and ignore them. Similarly, since $f(2t-1) \geq f(2t)$, Blue never claims more points than he would have as the hypothetical 'first player', and if $f(2t-1) > f(2t)$ Red can allocate the rest of Blue's move arbitrarily for him, and pretend that they have been claimed for the rest of game. Similarly if Blue every actually claims one of these points, Red can just allocate another point to him arbitrarily. Since the arbitrary extra points can only help Red, and since Φ was a winning strategy, Red wins the game, however this contradicts the assumption that Blue had a winning strategy . $\square$

Since the (n, ι) game is never a draw, for each n, either the first or second player must have a winning strategy. Croft [13] asked the question, how long does it take for that player to win?

The main result of this chapter is that neither player can win in time less than $(1 - o(1))n$. In fact, we prove a stronger result by considering the Maker-Breaker version of the game. A *Maker-Breaker game* is a pair $MB(X, \mathcal{F})$ where X is a set and $\mathcal{F} \subset \mathbb{P}(X)$, and as before we call X the board and the members $F \in \mathcal{F}$ winning sets. Two players, Maker and Breaker, alternately claim unclaimed points from the board, Maker colouring his points red and Breaker blue. If Maker is able to claim all points from a winning set he wins, otherwise Breaker wins. As before if Maker has a winning strategy we call $MB(X, \mathcal{F})$ a *Maker win*, and if Breaker has a winning strategy we call it a *Breaker win*. The *Maker-Breaker* (n, f) *game* is the Maker-Breaker game played on the same board with the same winning sets as the (n, f) game.

If we consider a positional game $(X, \mathcal{F})$ and the corresponding Maker Breaker game $MB(X, \mathcal{F})$, then if $MB(X, \mathcal{F})$ is a Breaker win it is clear that the second player in $(X, \mathcal{F})$ has a drawing strategy, and hence $(X, \mathcal{F})$ is a draw. However the converse is not true, there exists games where $(X, \mathcal{F})$ is a draw and yet $MB(X, \mathcal{F})$ is a Maker win. Similarly if $(X, \mathcal{F})$ is a first player win then $MB(X, \mathcal{F})$ is a Maker win, but again the converse is not true.

However in the case where $\mathcal{F}$ is n-regular, if Maker has a strategy to win $MB(X, \mathcal{F})$ in his first n moves, then by following the same strategy he can also win $(X, \mathcal{F})$. Following on from this simple observation Hefetz, Krivelevich and Szabó [28] considered a number of different games $MB(X, \mathcal{F})$, with $\mathcal{F}$ n-regular, in which it had been shown by Hefetz, Krivelevich, Stojaković and Szabó [27] that Maker has a winning strategy in his first $n + 1$ moves. By analysing these strategies closely they were able to show the games $(X, \mathcal{F})$ were a first player win. So, if we could show that in the Maker Breaker (n, ι) game Breaker has a strategy to delay Maker's win until time $(1 - o(1)n$ the same strategy could hopefully be adopted by the losing player in the (n, ι) game, and if we could show that Maker had a winning strategy in a reasonably quick time we might hope to be able to adapt it to a winning strategy in the (n, ι) game. For more on Maker-Breaker games see the monograph of Beck [7].

It is obvious that the Maker-Breaker (n, ι) game is a Maker win. We will consider the question of how long it takes for Maker to win. More formally given a strategy Φ for Breaker and a winning strategy Ψ for Maker, at some time $T(\Phi, \Psi)_n$ Maker will first fully occupy a winning set. We let $T_n = \max_\Phi \min_\Psi T(\Phi, \Psi)_n$, that is, T_n is the first time at which, with perfect play, Maker is guaranteed to have won. It is simple to see that $2\sqrt{n} - 1 \leq T_n \leq n$, the lower bound since before this time neither player has claimed n points.

In Section 4.2 we describe a simple strategy that gives a linear lower bound on T_n which is also applicable to a variation of the original n-in-a-row game. In Section 4.3 we show that, perhaps surprisingly, Breaker has a strategy that gives $T_n \geq (1 - o(1))n$.

Theorem 31. $T_n \geq n - o(\sqrt{n} \log n)$

This strategy will also give similar lower bounds for the ordinary (non Maker-Breaker) (n, ι) game. Whoever is the losing player can adopt this strategy and delay his loss, whether it is the first or second player will not affect the analysis of the strategy.

4.2 A weak pairing strategy

As we mentioned before, it is possible to show that the Maker-Breaker $(n, 1)$ game is a Breaker win for $n \geq 8$ by utilising a pairing strategy. A direct pairing strategy cannot be described for the Maker-Breaker (n, ι) game, since players claim more than one point at once. However in this section we are able utilise a similar idea to give a lower bound for T_n. Instead of pairing points, our plan is to assign to each point a direction, and have

Breaker's strategy to be as follows: for each point that Maker claims, Breaker claims the next unclaimed point in that direction. If Maker wants to fully occupy a line, say from East to West, then he cannot claim too many points in it that have been assigned the directions East or West, or Breaker will claim a point inside the line. So we aim to find a way to assign directions to points such that each winning set will have approximately the right number of each direction in it.

Theorem 32. $T_n \geq \frac{2}{11}n - 6$.

Proof. We define a function $f : \mathbb{Z}^2 \to \{\text{N, NE, E, SE, S, SW, W, NW}\}$ such that:

- the points $(1,1)$ to $(11,1)$ are mapped to N, NE, E, SE, S, SW, W, NW, N, NE, and E respectively,
- $f(x,y) = f(x+11,y)$ for all $x,y \in \mathbb{Z}$,
- $f(x,y+1) = f(x-3,y)$ for all $x,y \in \mathbb{Z}$.

So f is periodic with period 11 on $\{(x,y) : x \in \mathbb{Z}\}$ for any y, and we shift the pattern by 3 to go from (x,y) to $(x,y+1)$.

[Here the number 11 was chosen since we want a function that is periodic, with the same period, in each direction. If a function is periodic horizontally on $\mathbb{Z}^2$ and shifts by p to go from a row to the row above then it will clearly be periodic horizontally, vertically and diagonally, however it might have a smaller period. It is a simple check that to have the same period vertically it needs a period co-prime to p, and for the diagonals it needs a period co-prime to both $p-1$ and $p+1$. So for $p = 3$ we need a period co-prime to $2, 3, 4$, but also larger than 8, since each direction needs to appear at least once, and the smallest such number is 11.]

We think of this function as placing arrows in each square in the grid, for example Figure 4.2 shows the pattern on $[10]^2$, that is the bottom left square is $(1,1)$.

We have defined f in such a way that for any $x \in \{\text{N, NE, E, SE, S, SW, W, NW}\}$ and for any direction, if we look at 11 consecutive points in a row $v_1, \ldots, v_{11}$ in any of the eight directions then $1 \leq |\{v_i : f(v_i) = x\}| \leq 2$, that is, the number of points assigned to each direction is between 1 and 2. Breaker's strategy can now be described as follows: at time $2t$ Breaker looks at the $2t-1$ points Maker claimed on his turn $v_1, \ldots, v_{2t-1}$ and for each v_i claims the next available point in the direction $f(v_i)$ (any further points are claimed arbitrarily).

Suppose that Maker wins at time $2t+1$. We consider the n points in the winning line $L = \{v_1, \ldots, v_n\}$ just before Breaker moves at time $2t$. Without loss of generality we will

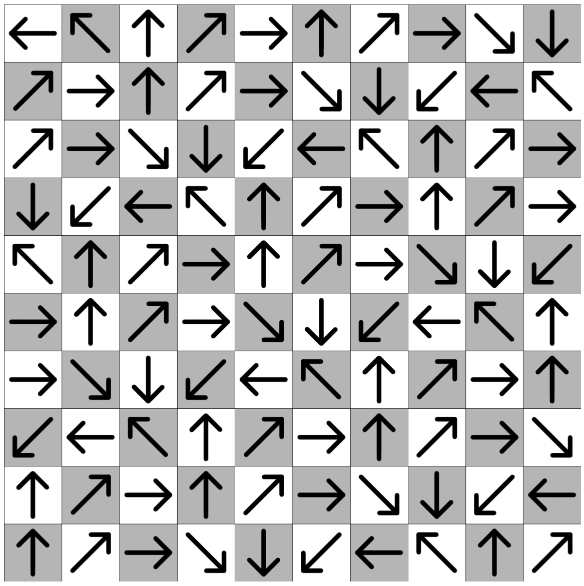

Figure 4.2: f on $[10]^2$.

assume L is in the East-West direction (the other cases can be treated similarly). If Maker has claimed any point $v_i \in L$ such that $f(v_i) = \mathrm{E}$ then Maker must also have claimed v_j for all $j > i$ since otherwise Breaker will claim one of then at time $2t$. Similarly if Maker has claimed any point v_i such that $f(v_i) = \mathrm{W}$ then Maker must also have claimed v_j for all $j < i$. Now if Maker has claimed 3 points $\{v_{i_1}, v_{i_2}, v_{i_3}\}$, $i_1 < i_2 < i_3$, such that $f(v_{i_j}) = \mathrm{E}$ for all j, then $|i_3 - i_1| > 11$ and hence there is some point v_k, $i_1 < k < i_3$, such that $f(v_k) =$W. By the preceding comment Maker must then already have claimed v_j for all $j > i_1$ and also for all $j < k$ and hence Maker must already have claimed the whole line at time $2t - 1$, contradicting our initial assumption.

Therefore Maker can only claim at most 4 of the points v_i such that $f(v_i) \in \{\mathrm{E,W}\}$ before time $2t + 1$. Of the n points in L at least $\frac{2}{11}(n - 10) \geq \frac{2}{11}n - 2$ of the v_i satisfy $f(v_i) \in \{\mathrm{E,W}\}$, therefore at least $\frac{2}{11}n - 6$ points in L must be unclaimed at time $2t + 1$. So $T_n = 2t + 1 \geq \frac{2}{11}n - 6$. $\square$

The constant $\frac{2}{11}$ could be improved by picking a larger prime instead of 11, and the same proof would show that $T_n \geq (\frac{1}{4} - \epsilon)n - C(\epsilon)$, where $C(\epsilon)$ is some constant depending on ϵ.

We also mention that the same strategy can be used to play a generalisation of the normal n-in-a-row game. One variation of positional games is that of biased positional games. In a *positional game* $(X, \mathcal{F})$ *with bias* (a, b) we have two players, Red and Blue, who takes turns claiming points of the board X, with the winner being the first to fully claim a winning set. However, now on Red's turn he claims a points, and on Blue's turn he claims b points. One can define biased Maker-Breaker games in a similar fashion. Unlike the normal Maker-Breaker n-in-a-row game, which is a Breaker win for $n \geq 8$, it is easy to see that the Maker-Breaker n-in-a-row game with bias $(a : b)$ is not a Breaker win for any value of n if $a > b$. Indeed Maker can play as follows: In his first $\frac{a^{n-1}}{(a-b)^{n-1}}$ turns Maker chooses points in $\frac{a^n}{(a-b)^{n-1}}$ distinct rows in $\mathbb{Z}^2$. During this time Breaker can only claim points in at most $\frac{ba^{n-1}}{(a-b)^{n-1}}$ of these rows. Therefore at least $\frac{a^{n-1}}{(a-b)^{n-2}}$ of these rows have none of Breaker's points in them. Maker now uses his next $\frac{a^{n-2}}{(a-b)^{n-2}}$ turns to try to claim a consecutive point in each of these rows. Since Breaker only plays $\frac{ba^{n-2}}{(a-b)^{n-2}}$ points during this period, at the end there are at least $\frac{a^{n-2}}{(a-b)^{n-3}}$ rows with two consecutive points claimed by Maker and none by Breaker. By repeating this argument, we end up with a row which contains n consecutive points of Maker's and none of Breaker's.

Csmiraz [14] showed that there is some constant c such that the Maker-Breaker n-in-a-row game with bias $(a : a)$ is a Breaker win for $n \geq ca^2 \log a$, and so also with bias $(a : b)$ for any $a \leq b$. Pluhár [43] improved this bound to $n \geq a + 80 \log_2 a + 160$ for $a \geq 1000$. Hsieh and Tsai [30] were able to prove a lower bound of $n \geq 4a + 7$ for all a and this was improved further by Chiang, Wub, and Lin [12] to $n \geq 3a + \phi(a) - 1$, for some logarithmic function ϕ. We note that the strategy above gives a simple argument for a linear lower bound.

Theorem 33. *In the Maker-Breaker n-in-a-row game with bias $(a : a)$, Breaker has a winning strategy if $n \geq \frac{11}{2}a + 33$.*

Proof. Breakers strategy is the same as in Theorem 32. On each turn Breaker looks at the a points Maker claimed on his last turn $v_1, \ldots, v_a$ and for each v_i claims the next available point in the direction $f(v_i)$.

Suppose that Maker wins at time t. We consider the n points in the winning line $L = \{v_1, \ldots, v_n\}$ just before Breaker moves at time t. Without loss of generality we will assume L is in the East-West direction (the other cases can be treated similarly). By the same argument as in Theorem 32 Maker can only claim at most 4 of the points v_i such that $f(v_i) \in \{\text{E,W}\}$ before time t. Of the n points in L at least $\frac{2}{11}(n - 10) \geq \frac{2}{11}n - 2$ of the v_i satisfy $f(v_i) \in \{\text{E,W}\}$, therefore at least $\frac{2}{11}n - 6$ points in L must be unclaimed at time t. Therefore if Maker wins at time t, $\frac{2}{11}n - 6 \leq a$, that is $n \leq \frac{11}{2}a + 33$. □

Again, by considering a more complicated pattern we could improve this bound to $n \geq (4 + \epsilon)a + C(\epsilon)$. The multiplicative constant in this bound is not an improvement on Chiang, Wub, and Lin or Pluhár's results, however the strategies in these two papers are quite complicated, both involving splitting the plane into small finite sub-boards, defining a strategy on each board, and then checking that both no winning lines are formed inside, or between sub-boards. In contrast this strategy is both simple, and a natural extension of the pairing strategy methods for the $(1:1)$ case. Furthermore it is possible that by picking a better pairing rule of this kind one could improve this bound.

4.3 Proof of Theorem 31

In this section we give a strategy for Breaker such that, for each $\delta > 0$, Maker cannot win before time $n - \delta\sqrt{n}\log n$, for large enough n, and hence show that $T_n \geq n - o(\sqrt{n}\log n)$. When we consider the board position 'at time t' we mean just prior to the turn where t points are claimed.

Proof of Theorem 31. Our first step is to cover $\mathbb{Z}^2$ with a family of lines of length $2n$. On every horizontal line $\{(x,i) : x \in \mathbb{Z}\}$ we take a line of length $2n$ starting at (jn, i) for each $j \in \mathbb{Z}$, that is the line

$$F_{i,j} = \{(x,i) : jn \leq x \leq (j+2)n - 1\}.$$

Note that every point $v \in \mathbb{Z}^2$ is in 2 such lines, and every horizontal winning set is a subset of one of these lines. We do the same for vertical and diagonal lines, that is we let

$$G_{i,j} = \{(i,y) : jn \leq y \leq (j+2)n - 1\},$$

$$H_{i,j} = \{(i+k,k) : jn \leq k \leq (j+2)n - 1\},$$

$$I_{i,j} = \{(i+k,-k) : jn \leq k \leq (j+2)n - 1\},$$

and take

$$\mathcal{L} = \bigcup_{i,j\in\mathbb{Z}} \{F_{i,j}\} \cup \bigcup_{i,j\in\mathbb{Z}} \{G_{i,j}\} \cup \bigcup_{i,j\in\mathbb{Z}} \{H_{i,j}\} \cup \bigcup_{i,j\in\mathbb{Z}} \{I_{i,j}\}.$$

Note that every point $v \in \mathbb{Z}^2$ is in 8 members of this family, and also every winning set is contained in a member of the family.

Given a line of length $2n$ with some points claimed but no winning set fully claimed, Breaker can place 2 more points inside that line such that no winning set can be fully claimed by Maker. So at time $2t$ Breaker can spoil t of these lines. Indeed given such a line, without loss of generality $L = \{(x,0) : 0 \leq x \leq 2n-1\}$, let $(v,0)$ be the largest point in $\{(x,0) : 0 \leq x \leq n-1\}$ which is either blue or unclaimed. Similarly let $(w,0)$ be the smallest point in $\{(x,0) : n \leq x \leq 2n-1\}$ which is either blue or unclaimed. Note that, since Maker does not have a winning set in L, it follows that $|v - w| < n$.

After Breaker claims $(v, 0)$ and $(w, 0)$ then Maker can no longer fully claim a winning set in L. After Breaker has played in such a way in $L \in \mathcal{L}$ we call L *bad*; otherwise L is *good*. Covering the winning sets with lines in this way allows us to simplify the analysis of this game by Breaker's strategy will be as follows: at time $2t$ he picks the t good $L \in \mathcal{L}$ which have the most red points in them and spoils them (any further points are claimed arbitrarily).

For a given play of the game we define

$$\mathcal{A}_r^t = \{L \in \mathcal{L} : L \text{ is good and the number of red points in } L \text{ at time } t \text{ is at least } r\}.$$

Suppose that Maker wins at time $t < n - \delta\sqrt{n}\log n$, for some $\delta > 0$. Then we must have that $|\mathcal{A}_{n-t}^t| > 0$. We claim that for all $C \leq \min\{\frac{n-t}{2\log n}, \frac{t}{8}\}$

$$|\mathcal{A}_{(n-t)-C\log n}^{t-2C}| > C\frac{t}{4} - C\frac{8t}{\log n}.$$

The claim clearly holds for $C = 0$, suppose it holds for a given value of $C < \min\{\frac{n-t}{2\log n}, \frac{t}{8}\}$. Then we must have

$$|\mathcal{A}_{(n-t)-C\log n}^{t-2C-1}| > (C+1)\frac{t}{4} - C\frac{8t}{\log n}$$

since Breaker will spoil $\frac{t-2C-1}{2} \geq \frac{t}{4}$ of the $L \in \mathcal{L}$ with his turn. We claim that now

$$|\mathcal{A}_{(n-t)-(C+1)\log n}^{t-2C-2}| > (C+1)\frac{t}{4} - (C+1)\frac{8t}{\log n}.$$

Indeed since each point is in 8 of the $L \in \mathcal{L}$ then by claiming $t \geq t - 2C - 2$ points Maker can only claim $\log n$ points in at most $8\frac{t}{\log n}$ sets. Therefore the claim holds for all $C \leq \min\{\frac{n-t}{2\log n}, \frac{t}{8}\}$.

Now if $\min\{\frac{n-t}{2\log n}, \frac{t}{8}\} = \frac{t}{8}$, then $t \leq \frac{4n}{\log n}$, so we conclude that with $C = \frac{t}{8}$

$$\begin{aligned}
|\mathcal{A}_{\frac{n}{4}}^{\frac{3t}{4}}| &\geq |\mathcal{A}_{(n-t)-\frac{t}{8}\log n}^{t-\frac{t}{4}}| \\
&> \frac{t^2}{2^5} - \frac{t^2}{\log n} \\
&= \Omega(t^2).
\end{aligned}$$

But now to claim at least $\frac{n}{4}$ points in at least $\Omega(t^2)$ of the $L \in \mathcal{L}$ requires at least $\frac{1}{8}\Omega(t^2)\frac{n}{4} = \Omega(nt^2)$ points. However by time $\frac{3t}{4}$ Maker has claimed at most $O(t^2)$ points, a contradiction.

Similarly in the case where $\min\{\frac{n-t}{2\log n}, \frac{t}{8}\} = \frac{n-t}{2\log n}$, we conclude that with $C = \frac{n-t}{2\log n}$,

$$\begin{aligned} |\mathcal{A}^{t-2C}_{\frac{(n-t)}{2}}| &= |\mathcal{A}^{t-2C}_{(n-t)-\frac{n-t}{2\log n}\log n}| \\ &> \left(\frac{n-t}{2\log n}\right)\left(\frac{t}{4}\right) - \left(\frac{n-t}{2\log n}\right)\left(\frac{8t}{\log n}\right) \\ &= \Omega\left(\frac{(n-t)t}{\log n}\right). \end{aligned}$$

But now to claim at least $\frac{(n-t)}{2}$ points in at least $\Omega\left(\frac{(n-t)t}{\log n}\right)$ of the $L \in \mathcal{L}$ requires at least $\frac{1}{8}\Omega\left(\frac{(n-t)t}{\log n}\right)\frac{(n-t)}{2} = \Omega\left(\frac{(n-t)^2 t}{\log n}\right)$ points. The minimum of $\frac{(n-t)^2 t}{\log n}$ for $t \in [2\sqrt{n}-1, n-\delta\sqrt{n}\log n]$ is at $t = n-\delta\sqrt{n}\log n$ and so at time $t-2C$ Maker must have claimed at least $\Omega(n^2\log n^2)$ points. However in the entire game Maker will claim at most $O(n^2)$ points, a contradiction. $\square$

The argument in the proof is similar in nature to a potential argument. We have assigned to each of the $L \in \mathcal{L}$ a measure of the danger it poses (the number of red points in it), and we have greedily tried to minimise this danger by playing in the most dangerous line at each stage. In fact Kane [36] showed that by picking a suitable potential function and using a similar strategy you can improve the bound to $T_n \geq n - O(\log(n))$. It would be interesting to know if this could be improved, or if a strategy for Maker can be found to prove a corresponding upper bound on T_n.

Question 34. *What is the exact value of T_n?*

In the proof of Theorem 31, once we had covered all the winnings sets with the lines $L \in \mathcal{L}$, Breaker's strategy only depended on the number of Maker's points in each L and, rather than trying to stop Maker claiming a winning set, Breaker was just trying to stop Maker claiming n points in some L unopposed. So we can think about the proof of Theorem 31 as analysing a simpler game. We have a set of bins $\{B_L : L \in \mathcal{L}\}$, on the $2t-1^{\text{th}}$ turn Maker gets to place at most $m(2t-1)$ balls spread across the bins as he chooses and then on the $2t^{\text{th}}$ Breaker gets to choose $b(2t)$ bins and remove them and the balls in them from the game. The game goes on for $2T$ turns and Maker wins if at the end he still has at least W balls in some remaining bin, otherwise Breaker wins.

Since each point in $\mathbb{Z}^2$ is in 8 $L \in \mathcal{L}$ we can think of a move of Maker's in the (n, ι) game where he claims $2t-1$ points as one in this new game where he places at most $8(2t-1)$ balls, since some points might be in lines Breaker has already spoilt. Similarly, in a move in the (n, ι) game where he claims $2t$ points, Breaker's strategy in Theorem 31 corresponds to choosing t bins and removing them and the balls in them from the game. The proof of Theorem 31 then corresponds to showing that, if Breaker follows his strategy, Maker can't win the ball bin game with $m(2t-1) = 8(2t-1)$, $b(2t) = t$, $2T \leq n - \delta\sqrt{n}\log n$ and $W = n-(2T+1)$. Since then, directly after Breakers move on the $2T^{\text{th}}$ turn in the (n, ι) game, Maker doesn't have $n-(2T+1)$ points in any good $L \in \mathcal{L}$ and so on his move he can't win.

The benefit of thinking about this ball bin game is that it is much simpler to analyse. It is not hard to find optimum strategies for both Maker and Breaker, and reduce the problem of who wins to the computation of a specific sum. In a joint work with Mark Walters, which is in preparation, we use this idea to study some generalisations of the (n, f) game. One generalization of the n-in-a-row game is to allow arbitrary slopes. That is we can play a Maker-Breaker game on $\mathbb{Z}^2$ where Maker wins if he can claim n consecutive points in a line, with *any* slope. Clearly this game is easier for Maker than the n-in-a-row game. Indeed Beck [7] showed that this game is a Maker win for all n, whereas, as mentioned in the introduction, the n-in-a-row game is a Breaker win for $n \geq 8$. Since this game is easier for Maker it raises the possibility that Maker can win the modified version of this game where the number of points picked on each turn is increasing in time less than $(1-o(1))n$.

Also, if we examine the proof of Theorem 31 we see that a similar conclusion would hold whenever $f(2t-1)$ and $f(2t)$ are both linear in t. Specifically if $f(2t-1) \leq c_1 t$ and $f(2t) \geq c_2 t$ for $c_1, c_2 > 0$, then the same calculation would show that Maker cannot win before the time at which he is playing $(1-o(1))n$ points in a turn. It is natural to consider whether or not a similar statement might be true for different growth rates of $f(2t-1)$ and $f(2t)$, in either the (n, f) game or the above generalisation. More explicitly if $f(2t-1) = \Theta(t^{\alpha})$ we can consider how large $f(2t)$ must be to delay Maker's win until the time at which Maker is playing $(1-o(1))n$ points in a turn.

We are able to show that, for the generalised version of the game with winning lines of arbitrary slope, if $f(2t-1) = \Theta(t^{\alpha})$ for $\alpha > 1$ then there exists some constant C such that if $f(2t) \geq C\log(t)$ then Breaker can delay Maker's win until the time at which Maker is playing $(1-o(1))n$ points in a turn. Similarly, in the case where $\alpha = 1$ Breaker can delay Maker's win if $f(2t) \geq C(\log(t))^3$. Conversely, if $f(2t-1) = \Theta(t^{\alpha})$ for any fixed $\alpha > 0$, then there is some constant C' such that if $f(2t) \leq C'\log(t)$ then there exists some $\epsilon > 0$ such that Maker can win, and in fact win in the sense of the (n, f) game, before the time at which he is playing $(1-\epsilon)n$ points in a turn. When $\alpha > 1$ these two bounds only differ by a constant factor, however when $\alpha = 1$ there is a still a gap. When $\alpha < 1$ our methods give a much worse bound that is still polynomial in t for the required growth rate of $b(t)$ to delay Maker's win.

As we mentioned in the introduction, since the analysis of the strategies in Theorem 32 and Theorem 31 would be unchanged, up to a small constant, if Breaker were to play first, both of these strategies can be used by the losing player in the (n, ι) game and so the lower bounds on T_n are also applicable to the (n, ι) game. Since the (n, ι) game cannot end in a draw, either the first or second player will have a winning strategy, Croft also asked:

Question 35. *Is the (n, ι) game a first or second player win?*

As mentioned in the introduction, a small case analysis shows that player 1 wins for $n = 1, 3, 4, 6, 7$ and player 2 wins for $n = 2, 5$. It is not clear if there is a simple formula that decides which player wins for general n.

The results in this chapter have been submitted for publication.

Chapter 5

Combinatorial derivations

5.1 Introduction

Many combinatorial problems related to subsets of the integers have natural generalizations to arbitrary groups. For example a number of problems in Ramsey theory are concerned with questions of the following type: Given a partition of $\mathbb{Z}^k$ into finitely many sets, must one of the sets contain a subset with certain structural properties? For instance Van der Waerden's Theorem [52] says that whenever we partition $\mathbb{Z}$ into finitely many sets one of the sets must contain arbitrarily large arithmetic progressions. When these properties make reference to the group structure of $\mathbb{Z}^k$, as in Van der Waerden's Theorem, it is natural to consider these problems in a more general setting, by replacing $\mathbb{Z}^k$ with an arbitrary infinite group G.

For example we say that a subset S of a group G is *symmetric* if for all $s \in S$ we have that $s^{-1} \in S$, that is $S = S^{-1}$. A subset S of a group G is *symmetric about a point* $g \in G$ if for all $s \in S$ we have that $gs^{-1}g \in S$, that is $S = gS^{-1}g$. So for example an arithmetic progression of length $2k+1$, $\{a, a+d, a+2d, \dots, a+2kd\}$, is symmetric about the point $a + kd$. Therefore Van der Waeden's theorem implies that whenever we partition $\mathbb{Z}$ into finitely many sets one of the sets must contain arbitrarily large sets which are symmetric about a point. Banakh and Protasov showed:

Theorem 36 (Banakh and Protasov [3]). *Given a partition $\mathbb{Z}^k = A_1 \cup A_2 \cup \ldots \cup A_k$ then there exists some i such that the set A_i contains an infinite set symmetric about a point $x \in \mathbb{Z}^k$. Conversely it is possible to partition $\mathbb{Z}^k = A_1 \cup A_2 \cup \ldots \cup A_{k+1}$ such that no A_i contains an infinite set symmetric about a point.*

For an arbitrary infinite group G we can define $\nu(G)$ to be the smallest k such that there exists a partition of G into k sets, none of which contain an infinite set symmetric around some point of G. Theorem 36 then says that $\nu(\mathbb{Z}^k) = k+1$. Banakh and Protasov [3] were able to calculate $\nu(G)$ for all abelian groups, however not much is known about $\nu(G)$ for about arbitrary infinite groups. In particular, for the free group on two generators, F_2, it is unknown whether $\nu(F_2)$ is even finite, although Gryshko and Khelif [25] were able to show that $\nu(F_2) > 2$.

Some results in this area are concerned with varying notions of the combinatorial size of subsets of an infinite group, see the survery [45]. Given a notion of size, a natural question to ask if, if we partition the group, or a subset of the group, into a finite number of sets, what can we say about the size of these sets? In this chapter we consider some problems of this type, as well as how the varying notions of combinatorial size relate to each other.

For a subset A of an infinite group G we denote

$$\Delta(A) = \{g \in G \,:\, |gA \cap A| = \infty\}.$$

This is sometimes called the *derivation* (or *combinatorial derivation*) of A. We note that $\Delta(A)$ is a subset of AA^{-1}, the difference set of A. It can sometimes be useful to consider $\Delta(A)$ as the elements that appear in AA^{-1} 'with infinite multiplicity'.

For example, consider the group $(\mathbb{Z}, +)$, that is, the integers under addition. Let $O = \{2n+1 \,:\, n \in \mathbb{Z}\}$ be the set of odd numbers, and $E = \{2n \,:\, n \in \mathbb{Z}\}$ the set of even numbers. We see that if m is odd, then $(m+O) \cap O = \emptyset$, and if m is even then $(m+O) \cap O = O$. Therefore $\Delta(O) = E$. As a further example suppose we have some finite set $F \subset \mathbb{Z}$ such that $F = -F$, then we can easily find some subset A of $(\mathbb{Z}, +)$ such that $\Delta(A) = F \cup \{0\}$. Indeed, suppose $F = \{f_1, f_2, \ldots, f_n\}$, we consider the set

$$A = \{10^{nm+1} + f_1, 10^{nm+2} + f_2, \ldots, 10^{n(m+1)} + f_n \,:\, m \in \mathbb{N}\} \cup \{10^m \,:\, m \in \mathbb{N}\}.$$

Clearly $f_i \in \Delta(A)$ for all i, but any other non-zero difference only appears a finite number of times, and so $\Delta(A) = F \cup \{0\}$. A similar construction would work for any countable F.

In [44] Protasov analysed a series of results on the subset combinatorics of groups (see the survey [45]) with reference to the function Δ, and asked a number of questions. In this chapter we present answers to some of those questions.

A subset A of G is said to be:

- *large* if there exists a finite subset F of G such that $FA = G$;

- Δ-*large* if there exists a finite subset F of G such that $F\Delta(A) = G$.

For example, as before in $(\mathbb{Z}, +)$, both O and E are large, since $\{0,1\}+O = \{0,1\}+E = \mathbb{Z}$. Similarly, since $\Delta(O) = \Delta(E) = E$, we have that they are both Δ-large. There are also various concepts of 'small' for subsets of groups. A subset A of G is said to be:

- *small* if $(G \setminus A) \cap L$ is large for every large subset L of G;
- *P-small* if there exists an injective sequence $(g_n)_1^\infty$ in G such that $g_iA \cap g_jA = \emptyset$ for all i, j;
- *almost P-small* if there exists an injective sequence $(g_n)_1^\infty$ in G such that $|g_iA \cap g_jA| < \infty$ for all i, j;
- *weakly P-small* if for every $n \in \mathbb{N}$ there exists distinct elements $g_1, g_2, \ldots, g_n$ in G such that $g_iA \cap g_jA = \emptyset$ for all i, j.

For example, again in $(\mathbb{Z}, +)$, the set $A = \{10^n, 10^n + n \,:\, n \in \mathbb{N}\}$ is almost P-small, since $|n+A\cap m+A| < \infty$ for every $n, m \in \mathbb{Z}$, however it is not P-small, since $n+A\cap m+A \neq \emptyset$ for any $n, m \in \mathbb{Z}$. For more on these concepts see [41]. For a subset A of G and a finite subset F it is easy to see that $\Delta(FA) = F\Delta(A)F^{-1}$. Therefore for abelian groups it is apparent that if A is large, with say $FA = G$, we have that $F^{-1}F\Delta(A) = \Delta(G) = G$, and so A is Δ-large. Protasov asked [44]:

Question 37. *Is every large subset of an arbitrary infinite group G Δ-large?*

Question 38. *Is every nonsmall subset of an arbitrary infinite group G Δ-large?*

As mentioned, $\Delta(A)$ is a subset of AA^{-1}. Banakh and Protasov showed:

Theorem 39 (Banakh and Protasov [4])**.** *Let G be an infinite group. Given a decomposition $G = A_1 \cup \ldots \cup A_n$ then there exists an i and a subset F of G such that $|F| \leq 2^{2^{n-1}-1}$ and $FA_iA_i^{-1} = G$.*

Protasov also asked whether a similar result could hold true for some $\Delta(A_i)$.

Question 40. *Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that, for any group G and any decomposition $G = A_1 \cup \ldots \cup A_n$, there exists an i and a subset F of G such that $G = F\Delta(A_i)$ and $|F| \leq f(n)$?*

A subset A of G is said to be *sparse* if for every infinite subset X of G, there exists a non-empty finite subset F of X such that $\bigcap_{g\in F} gA$ is finite. A subset A of G is said to be ∇-*thin* if either A is finite, or there exists an $n \in \mathbb{N}$ such that $\Delta^n(A) = \{e\}$, where Δ^n denotes the iterated application of Δ. Protasov also asked if these two concepts were related:

Question 41. *Is every ∇-thin subset of a group G sparse?*

Hindman [29] showed that whenever we decompose an infinite group into finitely many sets, $G = A_1 \cup A_2 \cup \ldots \cup A_n$, then there is some i and some infinite sequence of elements $g_1, g_2, \ldots \in G$ such that the set of finite products of this sequence is contained within A_i. That is, for all $k \in \mathbb{N}$ and $j_1 < j_2 < \ldots < j_k$, $g_{j_1} g_{j_2} \ldots g_{j_k} \in A_i$. Using this result Protasov showed that whenever we decompose an infinite group into finitely many sets, $G = A_1 \cup A_2 \cup \ldots \cup A_n$, such that for all i, $A_i = A_i^{-1}$ and $e \in A_i$, then there exists some i and an infinite subset X of G such that $X \subset A_i$ and $\Delta(X) \subset A_i$, in the case where every conjugacy class in G is finite. Protasov also asked if this were true for arbitrary groups:

Question 42. *Let G be an infinite group. Given a decomposition $G = A_1 \cup \ldots \cup A_n$, such that $A_i = A_i^{-1}$ and $e \in A_i$ for $i \in \{1, \ldots, n\}$, does there exist an i and an infinite subset X of G such that $X \subset A_i$ and $\Delta(X) \subset A_i$?*

In Section 5.2 we present answers to the four questions posed by Protasov. We say a subset A of G is *cofinite* if there exists a finite subset H of G such that $A = G \setminus H$. Our main result is:

Theorem 43. *Let G be an infinite group. Given a subset X of G such that there exists a finite subset F of G such that FX is cofinite, and a decomposition $X = A_1 \cup \ldots \cup A_n$, then there exists an i and a subset F' of G such that $|F'| \leq |F|(|F|+1)^{2^{n-1}-1}$ and $F'\Delta(A_i) = G$.*

This provides a positive answer to Question 37 and Question 40. We also show:

Theorem 44. *Let G be an infinite group, A a subset of G. Then if A is ∇-thin, then A is sparse.*

Answering Question 41. Finally we also show:

Theorem 45. *Let G be an infinite group. Given an infinite subset A of G and a countable subset X of $\Delta(A)$ such that $X = X^{-1}$ and $e \in X$, there exists an infinite subset Y of A such that $\Delta(Y) = X$.*

Since for all infinite sets A we have that $e \in \Delta(A)$, this provides a positive answer to Question 42. Indeed given $G = A_1 \cup \ldots \cup A_n$ as in Question 42, at least one of the A_i must be infinite and so by Theorem 45 there exists an infinite subset $Y \subset A_i$ such that $\Delta(Y) = e$. Since $e \in A_i$ by assumption this Y satisfies the conclusion of Question 42.

5.2 Results

We will start by considering Question 37.

Lemma 46. *Let G be an infinite group, A a subset of G. If there exists a finite subset F of G such that FA is cofinite, then $F\Delta(A) = G$.*

Proof. We would like to find some finite set $X = \{x_1, \ldots, x_k\}$, such that the set of translates $\{x_1A, x_2A \ldots, x_kA\}$ has the property that, for any $g \in G$, we must have that $|gA \cap x_iA| = \infty$ for some i. Then, for all $g \in G$ we would have that $|x_i^{-1}gA \cap A| = \infty$ for some i and so $x_i^{-1}g \in \Delta(A)$ and so $g \in X\Delta(A)$. Therefore we could conclude that $X\Delta(A) = G$.

Let $F = \{f_1, \ldots, f_k\}$. Since FA is cofinite, there exists some finite subset H of G such that $f_1A \cup \ldots \cup f_kA = FA = G \setminus H$. Therefore we see that for any $g \in G$ there must exist an i such that $|gA \cap f_iA| = \infty$. Hence F satisfies the property above, and so $F\Delta(A) = G$. □

We note that Question 37 follows from Lemma 46 as a simple corollary.

Corollary 47. *Let G be an infinite group, A a subset of G. If A is large, then A is Δ-large.* □

Moving on to Question 38, we can also use the same argument as in Lemma 46 to show that sets which are not almost P-small are Δ-large.

Theorem 48. *Let G be an infinite group, A a subset of G. Then if A is not almost P-small, then A is Δ-large.*

Proof. Take a maximal set $F = \{f_1, \ldots, f_k\}$ such that $|f_iA \cap f_jA| < \infty$ for all i, j. Such a set exists and is finite since A is not almost P-small. Then, for all $g \in G$ we must have that $|gA \cap f_iA| = \infty$ for some i, since F is maximal. Hence $f_i^{-1}g \in \Delta(A)$ and so $G = F\Delta(A)$. □

We note however that there do exist sets A which are not weakly P-small (and so also not P-small), but which are still not Δ-large.

Example 49. *Consider the group $(\mathbb{Z}, +)$. Let $A = \{10^n\, ,\, 10^n + n \,:\, n \in \mathbb{N}\}$. Clearly any translate of A has non-empty intersection with A, and so A cannot be weakly P-small. However $\Delta(A) = \{0\}$ since each difference only appears a finite number of times in A.*

It remains to show that sets which are not small are Δ-large. We will be able to show this using ideas that are used in answering Question 40.

To motivate the proof we first consider the case $n = 2$. Given a decomposition of G into two sets $A \cup B$ what does it mean if $\Delta(A) \neq G$? Well in that case we have some $g \in G$, $g \notin \Delta(A)$. Therefore there are only a finite number of $h \in G$ such that the group elements h and $g^{-1}h$ are both members of A, since each such h is in $gA \cap A$. Therefore there is some finite subset H of G such that for all $h \in G \setminus H$, either $h \in B$ or $g^{-1}h \in B$. But then we have that $\{e, g\}B = B \cup gB = G \setminus H$ and so, by Lemma 46, $\{e, g\}\Delta(B) = G$. This idea motivates the following lemma which will be key to answering Question 40.

Lemma 50. *Let G be an infinite group. Let X be a subset of G such that there exist finite subsets F, H_1 of G such that $FX = G \setminus H_1$. Then given a decomposition of X into two sets $X = A \cup B$, either $F\Delta(A) = G$ or there exists $g \in G$ and a finite subset H_2 of X such that $(gF \cup \{e\})B \supset X \setminus H_2$.*

Proof. Let $F = \{f_1, \ldots, f_k\}$. If $F\Delta(A) \neq G$, then there exists $g \in G$, $g \notin F\Delta(A)$, that is, $f_i^{-1}g \notin \Delta(A)$ for $i = 1, \ldots, k$. Now, as before, for each i there are only finitely many $h \in X$ such that both h and $f_i^{-1}gh \in A$. Also we claim that there are only finitely many h such that none of the group elements $f_1^{-1}gh, \ldots, f_k^{-1}gh$ are in X. Indeed, since if $F^{-1}gh \cap X = \emptyset$ then we have that $gh \cap FX = gh \cap G \setminus H_1 = \emptyset$ and so $h \in g^{-1}H_1$.

Therefore we have that there exists some finite subset H_2 of X such that for all $h \in X \setminus H_2$ and for all i, no pair h, $f_i^{-1}gh$ are both in A, and at least one of the group elements $f_i^{-1}gh$ is in X. Therefore we have that $B \cup g^{-1}FB \supset X \setminus H_2$. □

We note at this point that this lemma allows us to settle the final part of Question 38, whether or not a subset A of G which is not small, must be Δ-large.

Corollary 51. *Let G be an infinite group, A a subset of G. If A is not small, then A is Δ-large.*

Proof. If A is not small then there exists a large set L such that $(G \setminus A) \cap L$ is not large. Without loss of generality let us assume that $A \subset L$. Then $L = (L \setminus A) \cup A$. Since L is large there exists a finite subset F of G such that $FL = G$. Therefore, by Lemma 50, if $F\Delta(A) \neq G$, then there exists $g \in G$ and a finite subset H_2 of L such that $(gF \cup \{e\})(L \setminus A) \supset L \setminus H_2$. Therefore there is some finite subset H_3 of G such that $F(gF \cup \{e\})(L \setminus A) \supset G \setminus H_3$. However it is then clear that there exists some finite subset F' of G such that $F'(L \setminus A) = G$, however by assumption $L \setminus A$ was not large, and so $F\Delta(A) = G$. Therefore A is Δ-large. □

Protasov noted that the proofs of Lemma 50 and Corollary 51 carry over to a slightly more general setting. Given an arbitrary family $\mathcal{I}$ of subsets of G we can define a set $A \subseteq G$ to be *$\mathcal{I}$-large* if there exists some $I \in \mathcal{I}$ and some finite subset F of G such that $FA \cup I = G$. For example if $\mathcal{C}$ is the set of finite subsets then there exists some finite subset F of G such that FA is cofinite, if and only if A is $\mathcal{C}$-large. Similarly we say a set $A \subseteq G$ is *$\mathcal{I}$-small* if $(G \setminus A) \cap L$ is $\mathcal{I}$-large for every $\mathcal{I}$-large subset L of G. A proper family $\mathcal{I} \subsetneq \mathbb{P}(G)$ of subsets of a group is called an *ideal* if $\mathcal{I}$ is closed under taking subsets and finite unions. An ideal $\mathcal{I}$ of G is called *left-invariant* if $gI \in \mathcal{I}$ for all $g \in G$ and $I \in \mathcal{I}$.

Lemma 52. *Let G be an infinite group, $\mathcal{I}$ a left-invariant ideal of G. Let $X \subset G$ be $\mathcal{I}$-large, that is there exists $I \in \mathcal{I}$ such that $FX \cup I = G$. Then, given a decomposition of X into two sets $X = A \cup B$, either $F\Delta(A) = G$ or there exists $g \in G$ and $J \in \mathcal{I}$ such that $(gF \cup \{e\})B \cup J \supset X$.*

Proof. Let $F = \{f_1, \dots, f_k\}$. If $F\Delta(A) \neq G$, then there exists $g \in G$, $g \notin F\Delta(A)$, that is, $f_i^{-1}g \notin \Delta(A)$ for $i = 1, \dots, k$. Now, as before, for each i there are only finitely many $h \in X$ such that both h and $f_i^{-1}gh \in A$, let us denote by H_1 the set of all such h.

Also we claim that the set of h such that none of $f_1^{-1}gh, \dots, f_k^{-1}gh$ are in X is a subset of $g^{-1}I \in \mathcal{I}$. Indeed, since if $F^{-1}gh \cap X = \emptyset$ then we have that $gh \cap FX = gh \cap G \setminus I = \emptyset$ and so $h \in g^{-1}I$.

Therefore for all $h \in X \setminus (H_1 \cup g^{-1}I)$ we have that for some i, $f_i^{-1}gh \in X$ and at most one of the set $\{h, f_1^{-1}gh, \dots, f_k^{-1}gh\}$ is in A. Therefore $B \cup g^{-1}FB = X \setminus (H_1 \cup g^{-1}I)$ and so, since $\mathcal{I}$ is a left invariant ideal, there is some $J \in \mathcal{I}$ such that $(gF \cup \{e\})B \cup J \supset X$. $\square$

Corollary 53. *Let G be an infinite group, $\mathcal{I}$ a left-invariant ideal of G and A a subset of G. Then if A is not $\mathcal{I}$-small, then $\Delta(A)$ is $\mathcal{I}$-large.*

Proof. If A is not $\mathcal{I}$-small then there exists a $\mathcal{I}$-large set L such that $(G \setminus A) \cap L$ is not $\mathcal{I}$-large. Without loss of generality let us assume that $A \subset L$. Then $L = (L \setminus A) \cup A$. Since L is $\mathcal{I}$-large there exists a finite subset F of G and an $I_1 \in \mathcal{I}$ such that $FL \cup I_1 = G$. Therefore, by Lemma 52, if $F\Delta(A) \neq G$, then there exists $g \in G$ and an $I_2 \in \mathcal{I}$ such that $(gF \cup \{e\})(L \setminus A) \supset L \setminus I_2$.

But then $F(gF \cup \{e\})(L \setminus A) \supset FL \setminus FI_2 = G \setminus (FI_2 \cup I_1)$, that is there is an $I_3 \in \mathcal{I}$ such that $F(gF \cup \{e\})(L \setminus A) \cup I_3 = G$. But by assumption $L \setminus A$ was not $\mathcal{I}$-large, and so $F\Delta(A) = G$. Therefore $\Delta(A)$ is $\mathcal{I}$-large. $\square$

We can apply Lemma 50 inductively to prove the main result in this chapter.

Proof of Theorem 43. We induct on n. The case $n = 1$ follows from Lemma 46. Given that the result holds for all $k < n$, and given a subset X of G and finite subsets F, H_1 of G such that $FX = G \setminus H_1$ and a decomposition $X = A_1 \cup \ldots \cup A_n$. We have, by Lemma 50, that either $F\Delta(A_1) = G$, or there exists $g \in G$ and a finite subset H_2 of G such that $(gF \cup \{e\})(A_2 \cup \ldots \cup A_n) \supset X \setminus H_2$. We apply the induction hypothesis to the set $Y = (A_2 \cup \ldots \cup A_n)$. Note that there exists a finite subset H_3 of G such that $F(gF \cup \{e\})Y = G \setminus H_3$. Thus we have that there exists an i and a subset F' of G such that

$$\begin{aligned}|F'| &\leq |F|(|F|+1)\big(|F|(|F|+1)+1\big)^{2^{n-2}-1} \leq |F|(|F|+1)\big((|F|+1)^2\big)^{2^{n-2}-1} \\ &\leq |F|(|F|+1)(|F|+1)^{2.2^{n-2}-2} \leq |F|(|F|+1)^{2^{n-1}-1}\end{aligned}$$

and $F'\Delta(A_i) = G$. □

It is known [4] that given a decomposition $G = A_1 \cup \ldots \cup A_n$, there exists an i and a subset F of G such that $|F| \leq 2^{2^{n-1}-1}$ and $FA_iA_i^{-1} = G$. Noting that $\Delta(A) \subset AA^{-1}$, Protasov [44] asked whether a similar result would hold true for some $\Delta(A_i)$. The following Corollary strengthens that result, answering Question 40.

Corollary 54. *Let G be an infinite group. Given a decomposition $G = A_1 \cup \ldots \cup A_n$ then there exists an i and a subset F of G such that $|F| \leq 2^{2^{n-1}-1}$ and $F\Delta(A_i) = G$.*

Proof. We apply Theorem 43 with $X = G$ and $F = \{e\}$ to get the result stated. □

Recently Banakh, Ravsky, and Slobodianiuk [5] strengthened Corollary 54 to show that there must exist some $F \subset G$ such that $F\Delta(A_i) = G$ and $|F| \leq \phi(n)$ where ϕ is some function which grows faster than any exponential function c^n, but slower than $n!$.

Theorem 43 says that if we decompose any large set into a finite number of pieces, at least one of the parts must be Δ-large. However there do exist Δ-large sets which decompose into two sets which are not Δ-large.

Example 55. *Consider the group $(\mathbb{Z}, +)$. Let $A = \{10^n : n \in \mathbb{N}\}$ and let*

$$B = \{10^1+1\} \cup \{10^2+1\} \cup \{10^3+2\} \cup \{10^4+1\} \cup \{10^5+2\} \cup \{10^6+3\} \cup \{10^7+1\} \ldots$$

where the pattern of numbers after the powers of ten will be $\{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots\}$, so that each $x \in \mathbb{N}$ appears an infinite number of times. Then if $X = A \cup B$ we see immediately that $\Delta(X) = \mathbb{Z}$, but $\Delta(A) = \Delta(B) = \{0\}$.

There also exist decompositions of large sets into a finite number of sets, none of which are large.

Example 56. *Consider the free group on 2 elements, $F(a,b)$. If we denote by aSb the set of reduced words in $F(a,b)$ that start with a and end with b, then it is clear that aSb is not large. Indeed no finite set of translates can contain the words a^n for all n. However we can decompose $F(a,b)$ as*

$$F(a,b) \setminus \{e\} = \bigcup_{x,y \in \{a,a^{-1},b,b^{-1}\}} xSy,$$

none of which are large.

We now consider Question 41.

Proof of Theorem 44. We will prove that every set which is not sparse is not ∇-thin. Given $A \subset G$ which is not sparse then there exists some infinite subset X of G such that for every finite subset F of X, $\bigcap_{g \in F} gA$ is infinite. In particular for every pair $g_i, g_j \in X$ we have that $|g_i A \cap g_j A| = |g_j^{-1} g_i A \cap A| = \infty$ and so $X^{-1}X \subset \Delta(A)$.

However for any infinite set X we claim that $X^{-1}X \subset \Delta(X^{-1}X)$. Indeed given $g_i^{-1} g_j \in X^{-1}X$ we have that

$$|g_i^{-1} g_j X^{-1} X \cap X^{-1}X| = |g_j X^{-1} X \cap g_i X^{-1} X| \geq |X \cap X| = |X| = \infty,$$

and so $g_i^{-1} g_j \in \Delta(X^{-1}X)$. Therefore if A is not sparse we have that $X^{-1}X \subset \Delta^n(A)$ for all $n \in \mathbb{N}$, and so $\Delta^n(A) \neq \emptyset$ for any n. □

We say a subset A of an infinite group G is n-sparse if for every infinite subset X of G there exists a subset F of X such that $|F| = n$ and $\bigcap_{g \in F} gA$ is finite. The proof of Theorem 44 shows in fact that every ∇-thin set is 2-sparse. However the converse is not true, there exist 2-sparse sets which are not ∇-thin.

Example 57. *Consider the group $(\mathbb{Z}, +)$. Let $A_1 = \{10^n : n \in \mathbb{N}\}$ and let*

$$A_2 = \{10^1 + 1\} \cup \{10^2 + 1\} \cup \{10^3 + 3\} \cup \{10^4 + 1\} \cup \{10^5 + 3\} \cup \{10^6 + 5\} \cup \{10^7 + 1\} \ldots$$

where the pattern of numbers after the powers of ten will be $\{1, 1, 3, 1, 3, 5, 1, 3, 5, 7, \ldots\}$, so that each odd $x \in \mathbb{N}$ appears an infinite number of times.

Then if $A = A_1 \cup A_2$ we see that $\Delta(A) = \{0\} \cup \{2n+1 : n \in \mathbb{Z}\}$, and so $\Delta^n(A) = \{2n : n \in \mathbb{Z}\}$ for all $n \geq 2$. However, given any infinite subset X of G we must have two *numbers a and b in X whose difference is even. Then $aA \cap bA$ is finite, since there are only a finite number of even differences in A less than any given number.*

Finally we turn to Question 42. In [44] it is shown that for all infinite groups G, and all subsets A of G such that $A = A^{-1}$ and $e \in A$, there exists some subset X of G such that $\Delta(X) = A$. Using a similar construction we are able to prove Theorem 45.

Proof of Theorem 45. Let $X = \{x_1, x_2, \ldots\}$, repeating the last element if X is finite, and let $Z_0 = Y_0 = \emptyset$. We define a new sequence. For all $n \in \mathbb{N}$ let

$$w_i = x_{i-\frac{n(n-1)}{2}} \quad \text{for} \quad \frac{n(n-1)}{2} + 1 \leq i \leq \frac{n(n+1)}{2}.$$

That is, $w_1 = x_1$, $w_2 = x_1, w_3 = x_2$, $w_4 = x_1, w_5 = x_2, w_6 = x_3$, $w_7 = x_1 \ldots$. Our plan is to add pairs of elements to Y such that each pair has difference w_i, but introduces no new differences with the elements already in Y. Given $Y_{i-1} = \bigcup_{j=0}^{i-1} Z_j$ we want to inductively find $Z_i = \{z_i, w_i z_i\} \subset A$ such that $(Z_i Y_{i-1}^{-1} \cup Y_{i-1} Z_i^{-1}) \cap Y_{i-1} Y_{i-1}^{-1} = \emptyset$. Equivalently Z_i needs to avoid the finite set $Y_{i-1} Y_{i-1}^{-1} Y_{i-1}$. This is always possible since $w_i \in \Delta(A)$ and so the number of such pairs is infinite. We let $Y = \bigcup_{i=1}^{\infty} Y_i$ and see that $\Delta(Y) = X$ as claimed. □

We note at this point that, perhaps surprisingly, it is necessary for the subset X in Theorem 45 to be countable.

Proposition 58. *There exists a group G, a subset A of G, and a subset Y of $\Delta(A)$ such that there does not exist any subset X of A with $\Delta(X) = Y$.*

Proof. Let $\kappa = \left(2^{\aleph_0}\right)^+$ (so assuming CH we would have that $\kappa = \aleph_2$), and let α be the initial ordinal of cardinality κ. Consider the group $G = (\mathbb{Z}_2)^\alpha$. That is, G is the direct product of κ copies of $\mathbb{Z}_2$. Let x_i, for $i \leq \alpha$, be be the element which is 1 in the i^{th} copy of $\mathbb{Z}_2$ and 0 elsewhere. Consider the set

$$A = \{x_n + x_i \,:\, 1 \leq n < \omega \,,\, \omega \leq i \leq \alpha\} \cup \{x_n \,:\, 1 \leq n < \omega\}.$$

We see that the set $X = \{x_i \,:\, \omega \leq i \leq \alpha\}$ is a subset of $\Delta(A)$, however we claim that there does not exist any subset Y of A such that $\Delta(Y) = X$. Indeed, suppose such a subset exists. Let i be such that $\omega \leq i \leq \alpha$. Since $x_i \in \Delta(Y)$ we have that the set of $n < \omega$ such that both x_n and $x_n + x_i$ are in Y is infinite. Let us call this set L_i. Since $2^{\aleph_0} < \kappa$ we must have that there exist $i, j \leq \alpha$ such that $L_i = L_j$. But then we also have that $x_i + x_j \in \Delta(Y)$, contradicting our initial assumption. □

This phenomenon arises since in the definition of the function Δ we only require that the intersection $|gA \cap A|$ is infinite. If instead we were to consider a more general function

$$\Delta_{|G|}(A) = \{g \in G \,:\, |gA \cap A| = |G|\},$$

an analogous version of Theorem 45 could be proved, by the same argument, for sets of larger cardinality. Indeed much of the work in this chapter, including more general versions of Lemma 46 and Theorem 43, can be easily adapted to state results in this framework. Given Theorem 45 we can answer Question 42, albeit in a slightly trivial way.

Corollary 59. *Let G be an infinite group, $G = A_1 \cup \ldots \cup A_n$, $A_i = A_i^{-1}$, $e \in A_i$ for $i \in \{1, \ldots, n\}$. Then there exists an i and an infinite subset X of G such that $X \subseteq A_i$ and $\Delta(X) \subseteq A_i$.*

Proof. By Theorem 45 we have that, within any infinite set A there exists an infinite subset X of A with $\Delta(X) = \{e\}$. At least one of the A_i must be infinite, and therefore such an X satisfies the statement. □

Corollary 54 says that if an infinite group G is split into a finite number of sets, then one of those sets must be Δ-large. For groups of larger cardinality can we prove similar results? For example:

Question 60. *Let G be an infinite group, with $|G| = \kappa$. Given $\mu < \kappa$, $|I| = \mu$, and a decomposition $G = \bigcup_{i \in I} A_i$, can we find a 'small' subset F of G such that $F\Delta(A_i) = G$?*

It is not true that we can always take F to be finite. In fact Protasov and Slobodianiuk [46] showed that any group G of regular cardinality can be decomposed into a countable number of subsets $G = \bigcup_{i=1}^{\infty} A_i$ such that for each i, $\mathrm{Cov}(A_i A_i^{-1})$=$|G|$, where $\mathrm{Cov}(A_i A_i^{-1})$ is $\min\{|X| \,:\, X A_i A_i^{-1} = G\}$ (note that $\Delta(A) \subset A_i A_i^{-1}$). They conjectured that this was true in fact for groups of arbitrary cardinality, and proved this conjecture for abelian groups in particular.

The results in this section are in preparation.

Bibliography

[1] B. Alspach, J.-C. Bermond, and D. Sotteau. Decomposition into cycles I: Hamilton decompositions. *Proceedings of the NATO Advanced Research Workshop on Cycles and Rays*, pages 9–18, 1990.

[2] D. Anick and M. Ramras. Edge decompositions of hypercubes by paths. *Pre-print*, 2013. `http://arxiv.org/abs/1308.4949`.

[3] T. Banakh and I. Protasov. Asymmetric partitions of abelian groups (in russian). *Matematicheskie Zametki*, 66(1):10–19, 1999.

[4] T. Banakh and I.V. Protasov. Ball structure and colorings of groups and graphs. *Math. Stud. Monorg. Ser*, 11, 2003.

[5] T. Banakh, O. Ravsky, and S. Slobodianiuk. On partitions of g-spaces and g-lattices. `http://http://http://arxiv.org/abs/1303.1427`, 2013. preprint, submitted to IJAC.

[6] B. Barden, J. Davis, R. Libeskind-Hadas, and W. Williams. On edge-disjoint spanning trees in hypercubes. *Information Processing Letters*, 70(1):13–16, 1999.

[7] J. Beck. *Combinatorial Games: Tic-Tac-Toe theory.* Cambridge University Press, 2008.

[8] Abu Zakariya Yahya ben Ibrahim al Hakim. Nuzhat al-arbab al-'aqulfi'sh-shatranj al-manqul. Manuscript, 1350.

[9] R.L. Berlekamp, J.H. Conway, and R.K. Guy. *Winning Ways for your mathematical plays, Vol. 2.* Academic Press, New York, 1982.

[10] E.D. Bryant, S. El-Zanati, C.V. Eynden, and D.G. Hoffman. Star decompositions of cubes. *Graphs and Comb.*, 17:55–59, 2001.

[11] G. Cairns. Pillow chess. *Mathematics Magazine*, 75(3):173–186, 2002.

[12] S. Chiang, I. Wub, and P. Lin. Drawn k-in-a-row games. *Theoretical Computer Science*, 412(35):4558–4569, 2011.

[13] H.T. Croft. personal communication, 2011.

[14] L. Csmiraz. On a combinatorial game with an application to go-moku. *Discrete Mathematics*, 29(1):19–23, 1980.

[15] P. Cull and D. J. Curtins. Knight's tours revisited. *Fibonacci Quarterly*, 16:276–285, 1978.

[16] T. R. Dawson. *L'Echiquier*, page 985 and 1054, 1928.

[17] J. DeMaio. Which chessboards have a closed knight's tour within the cube? *The Electronic Journal of Combinatorics*, 14, 2007.

[18] J. DeMaio and B. Mathew. Which chessboards have a closed knight's tour within the rectangular prism? *The Electronic Journal of Combinatorics*, 18, 2011.

[19] H. E. Dudeny. *Amusements in Mathematics*. Dover, 1958.

[20] S. El-Zanati and C.V. Eynden. Cycle factorizations of cycle products. *Discrete Mathematics*, 189(1-3):267–275, 1988.

[21] J. Erde, B. Golénia, and S. Golénia. The closed knight tour problem in higher dimensions. *The Electronic Journal of Combinatorics*, 19(4), 2012.

[22] L. Euler. Solution d'une question curieuse que ne paroît soumise á aucune analyse. *Histoire de l'Académie Royale des Sciences et des Belles-Lettres de Berlin*, 15:310–337, 1759.

[23] J. F. Fink. On the decomposition of n-cubes into isomorphic trees. *Graph Theory*, 14(4):405–411, 1990.

[24] M. Frolow. *Les Carrés Magiques, Plate VII.* 1886.

[25] Y. Gryshko and a. Khelif. Monochrome symmetric subsets of colored groups. *Journal of Combinatorial Theory, Series A*, 112:212–221, 2005.

[26] P. Hall. On representatives of subsets. *Journal of the London Mathematical Society*, 10(1):26–30, 1935.

[27] D. Hefetz, M. Krivelevich, Stojaković, and T. Szabó. Fast winning strategies in positional games. *European Conference on Combinatorics, Graph Theory and Applications*, 29:213–217, 2007.

[28] D. Hefetz, M. Krivelevich, and T. Szabó. Fast winning strategies in maker-breaker games. *Journal of Combinatorial Theory, Series B*, 99(1):39–47, 2009.

[29] N. Hindman and D. Strauss. *Algebra in the Stone-Čech compactification.* Walter de Grueters, San Mateo, California, 1998.

[30] M. Y. Hsieh and S. Tsai. On the fairness and complexity of generalized k-in-a-row games. *Theoretical Computer Science*, 285(1-3):88–100, 2007.

[31] M. S. Jacobson, M. Truszczyński, and Z. Tuza. Decompositions of regular bipartite graphs. *Discrete Mathematics*, 89(1):17–27, 1991.

[32] G. Jellis. Leapers at large. `http://www.mayhematics.com/t/pl.htm`. Accessed: 2014-01-08.

[33] G. Jellis. Theory of leapers. *Chessics*, 2(24):86–98, 1985.

[34] G. Jellis and T. H. Willcocks. The five free leapers. *Chessics*, 1(2):4–5, 1976.

[35] N. Kamčev. Generalised knight's tours. `http://arxiv.org/abs/1311.4109`, 2013. preprint.

[36] D. Kane. personal communication, 2012.

[37] D. E. Knuth. Leaper graphs. *Mathematical Gazette*, 78:274–297, 1994.

[38] M. Kraitchik. *Mathematical Recreations.* Dover, 1953.

[39] F. Leighton. *Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes.* M. Kaufmann Publishers, San Mateo, California, 1992.

[40] D.E. Lucas. *Recréations Mathématiques.* Gautheir Villars, Paris, 1892.

[41] Ie. Lutsenko and I.V. Protasov. Sparse, thin and other subsets of groups. *International Journal of Algebra and Computation*, 19:491–510, 2009.

[42] M. Mollard and M. Ramras. Edge decompositions of hypercubes by paths and by cycles. *Pre-print*, 2013. `http://arxiv.org/abs/1205.4161v3`.

[43] A. Pluhár. The accelerated k-in-a-row game. *Theoretical Computer Science*, 270(1-2):865–875, 2002.

[44] I.V. Protasov. Combinatorial derivation. *Manuscript*, 2011.

[45] I.V. Protasov. Selective survey on subset combinatorics of groups. *Journal of Mathematical Sciences*, 174(4):486–514, 2011.

[46] I.V. Protasov and S. Slobodianiuk. A note on partitions of groups. *Manuscript*, 2013.

[47] Y. Qing and J. J. Watkins. Knight's tours for cubes and boxes. *Congressus Numerantium*, 181:41–48, 2006.

[48] M. Ramras. Symmetric edge-decompositions of hypercubes. *Graphs and Combinatorics*, 7:65–87, 1991.

[49] G. Ringel. Über drei kombinatorische probleme am n-dimensionalen würfel und würfelgitter. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 20:10–19, 1954.

[50] A. J. Schwenk. Which rectangular chessboards have a knight's tour? *Mathematics Magazine*, 64(5):325–332, 1991.

[51] I. Stewart. Solid knight's tours. *Journal of Recreational Mathematics*, 4(1), 1971.

[52] B.L. Van der Waerden. Beweis einer baudetschen vermutung. *Nieuw Archief voor Wiskunde*, 15:212–216, 1927.

[53] S. Wagner and M. Wild. Decomposing the hypercube Q_n into n isomorphic edge-disjoint trees. *Discrete Mathematics*, 312(10):1819–1822, 2012.

[54] J. J. Watkins. Knight's tours on a torus. *Mathematics Magazine*, 70(3):175–184, 1997.

[55] J. J. Watkins. Knight's tours on cylinders and other surfaces. *Congressus Numerantium*, 143:117–127, 2000.

[56] T. G. L. Zetters. 8 (or more) in a row. *The American Mathematical Monthly*, 87(7):575–576, 1980.